10th Grade Maths Course

Year 2026

MAURER Stéfan

10th Grade Mathematics Syllabus

Table of Contents

ALGEBRA

ANALYSIS (CALCULUS)

GEOMETRY

STATISTICS & PROBABILITY

FUNDAMENTAL CALCULATION RULES

I - Subtraction

Definition 1 : The **opposite** of the number a is the number denoted $-a$ such that : $a+(-a)=0$.

Remarque : In this case, the sign "$-$" is not the mark of a negative number. Indeed, according to the definition, the opposite of the number -2 is the number $-(-2)$. However, $-2+2=0$, so we deduce that $-(-2)=2$ is a positive number.

Definition 2 : The **subtraction** of a by b is the operation that adds the number a and the opposite of the number b.

$$a-b=a+(-b)$$

Application : Group terms in order to simplify an expression.

$$\begin{aligned} 5+3x-6-5x &= 5+3x+(-6)+(-5x) \\ &= 3x+(-5x)+5+(-6) \\ &= -2x+(-1) \\ &= -2x-1 \end{aligned}$$

II - Multiplication

Property 1 : Multiplication takes precedence over addition (and subtraction).

- Multiplication is distributive over addition (and subtraction).
- $k\times(a+b)=k\times a+k\times b=ka+kb$
- $(a+b)(c+d)=a\times c+a\times d+b\times c+b\times d$

Application : Expand and simplify an expression.

$$\begin{aligned} 3x+5-2(x+1) &= 3x+5+(-2)\times(x+1) \\ &= 3x+5+(-2)\times x+(-2)\times 1 \\ &= 3x+5+(-2x)+(-2) \\ &= x+3 \end{aligned}$$

$$\begin{aligned} x-(2x+1) &= x+(-1)\times(2x+1) \\ &= x+(-1)\times 2x+(-1)\times 1 \\ &= x+(-2x)+(-1) \\ &= -1x+(-1) \\ &= -x-1 \end{aligned}$$

III - Division

Definition 3 : The **reciprocal** (or multiplicative inverse) of the number $a\neq 0$ is the number denoted $\frac{1}{a}$ such that : $a\times\frac{1}{a}=1$.

Remarque : The reciprocal of the number $\frac{a}{b}$ ($a\neq 0$ and $b\neq 0$) is the number $\frac{b}{a}$. Indeed, $\frac{a}{b}\times\frac{b}{a}=\frac{a\times b}{b\times a}=1$.

Property 2 : For any number a, $a\times 0=0\times a=0$.

Consequence : The number 0 has no reciprocal.

Démonstration : Prove that 0 has no reciprocal.

Proof by contradiction : If $\frac{1}{0}$ exists, then according to definition 3, $0 \times \frac{1}{0} = 1$.

However, according to property 2, for any number a, $0 \times a = 0$. Therefore, in particular, if $a = \frac{1}{0}$ we obtain $0 \times \frac{1}{0} = 0$. Thus we obtain that $0 = 1$. This result is absurd, so the initial assumption is false.

Conclusion : $\frac{1}{0}$ does not exist.

Definition 4 : The **division** of a by b ($b \neq 0$) is the operation that multiplies the number a and the reciprocal of the non-zero number b.

$$a \div b = a \times \frac{1}{b}$$

Consequence : The number 0 has no reciprocal, therefore division by 0 is impossible.

IV - Fractions

Property 3 :

- For any number $a \neq 0$, $-\frac{1}{a} = \frac{-1}{a} = \frac{1}{-a}$
- For all numbers $b \neq 0$, $c \neq 0$, $\frac{a}{b} = \frac{a \times c}{b \times c}$
- For all numbers $b \neq 0$, $d \neq 0$, $\frac{a}{b} + \frac{c}{d} = \frac{a \times d}{b \times d} + \frac{b \times c}{b \times d} = \frac{a \times d + b \times c}{b \times d}$
- For all numbers $b \neq 0$, $d \neq 0$, $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$. In particular, $a \times \frac{b}{c} = \frac{a}{1} \times \frac{b}{c} = \frac{a \times b}{c}$
- For all numbers $b \neq 0$, $c \neq 0$, $d \neq 0$, $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} / \frac{c}{d} = \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \times \frac{d}{c}$

V - Powers

Definition 5 : For any number a different from 0 and any natural integer n, we define the number $\boldsymbol{a^n}$:

If $n = 0$ then $a^0 = 1$
If $n = 1$ then $a^1 = a$
If $n > 1$ then $a^n = a \times a \times \cdots \times a$ (n factors of a)

Convention : The exponent always takes precedence over other operators.

Definition 6 : For any number a different from 0 and any relative integer n, we define the number $\boldsymbol{a^{-n}}$:

$$a^{-n} = \frac{1}{a^n}$$

That is to say, a^{-n} is the reciprocal of a^n.

Property 4 : For all numbers a and b different from 0, for all relative integers m and n :

- $a^m \times a^n = a^{m+n}$
- $\frac{a^m}{a^n} = a^{m-n}$
- $(a^m)^n = a^{m \times n}$
- $a^n \times b^n = (a \times b)^n$
- $\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n$

SETS OF NUMBERS

In mathematical language, a set of numbers is denoted using curly brackets (or square brackets when it comes to an interval, as we will see later in this course).
If the set has a name, then the curly brackets are preceded by the name of the set followed by the "= " symbol.

Definition 1 : We call $\mathbb{N}$ **the set of natural numbers** : $\mathbb{N} = \{0, 1, 2, 3, \dots\}$.

Remarque : The 3 dots indicate that the set contains an infinity of numbers that continue the same logical sequence (in definition 1, to go from one number to the next, we add 1).

Definition 2 : We call $\mathbb{Z}$ **the set of integers** (or relative integers) : $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$.

Definition 3 : We call $\mathbb{D}$ **the set of decimal numbers** : $\mathbb{D} = \left\{\frac{n}{10^p}, n \in \mathbb{Z}, p \in \mathbb{N}\right\}$.

Application : Write a decimal number as a decimal fraction.

$$1.732 = \frac{1.732}{1} = \frac{1732}{1000} = \frac{1732}{10^3}$$

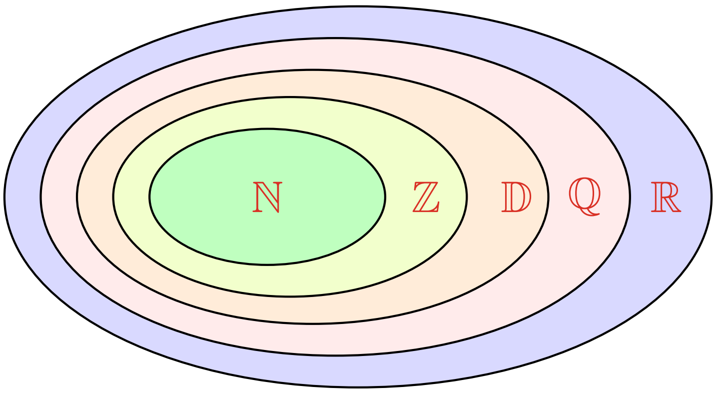
Remarque : Definition 3 specifies that a decimal number has a finite number of decimal places.

Definition 4 : We call $\mathbb{Q}$ **the set of rational numbers** : $\mathbb{Q} = \left\{\frac{n}{d}, n \in \mathbb{Z}, d \in \mathbb{N}, d \neq 0\right\}$.

Remarque : Rational numbers that are not decimals, written in "decimal form", have an infinite number of decimal places. However, these decimals follow certain rules, meaning that the same digits are found periodically.

$$\frac{1}{3} = 0.33\dots \qquad \frac{2}{11} = 0.1818\dots \qquad \frac{3}{7} = 0.428571428571\dots$$

Definition 5 : We call $\mathbb{R}$ **the set of real numbers**. $\mathbb{R} = \{x, x^2 \geq 0\}$



Remarque : Real numbers that are not rational are called irrational; $\sqrt{2}$ and π are irrational.

Notation : If a set E is contained in a set F, we say that E is included in F, and we denote it : $E \subset F$.

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{D} \subset \mathbb{Q} \subset \mathbb{R}$$

- To indicate that a number belongs to a set, we use $\in$: $5 \in \mathbb{N}, -12 \in \mathbb{Z}$.
- To indicate that a number does not belong to a set, we use $\notin$: $-6 \notin \mathbb{N}, \frac{3}{4} \notin \mathbb{Z}$.

ARITHMETIC IN $\mathbb{N}$

Definition 1 : To perform the **Euclidean division** of a natural number a by a strictly positive integer b means finding two natural numbers called **quotient** q and **remainder** r such that :

$$a = b \times q + r \quad \text{with} \quad 0 \leq r < b$$

Definition 2 : We say that the natural number b is a **divisor** of the natural number a when the remainder of the Euclidean division of a by b is equal to zero, meaning that there exists a natural number q such that :

$$a = b \times q$$

Vocabulaire : The following sentences have the same meaning.

- b is a divisor of a
- b divides a
- a is a multiple of b
- a is divisible by b

Démonstration : Prove that $\frac{1}{3} \notin \mathbb{D}$.

Proof by contradiction.

If $\frac{1}{3} \in \mathbb{D}$ then there exists a relative integer n and a strictly positive natural number p such that $\frac{1}{3} = \frac{n}{10^p}$.
So $3n = 10^p = 10 \times 10 \times \cdots \times 10$ (p factors)
So 3 is a divisor of 10^p.
This result is absurd so the initial assumption is false.
Therefore $\frac{1}{3} \notin \mathbb{D}$.

Definition 3 : We say that a **natural number** is **prime** when it has exactly 2 divisors : 1 and itself.

Definition 4 : We say that **natural numbers** are **coprime** (or mutually prime) when their only common divisor is 1.

Definition 5 : We say that a **natural number** a is **even** when it is divisible by 2, meaning there exists a natural number q such that : $a = 2 \times q$

Definition 6 : We say that a **natural number** a is **odd** when it is not divisible by 2, meaning there exists a natural number q such that : $a = 2 \times q + 1$

Property 1 : Let a be a natural number ;

a^2 is even (resp. odd) if and only if a is even (resp. odd).

Démonstration :

- If a is even then there exists a natural number q such that $a = 2 \times q$.
  So $a^2 = (2 \times q)^2 = 2^2 \times q^2 = 2 \times (2q^2)$, so a^2 is even.
- To prove that if a^2 is even then a is even, we use the **contrapositive** of this proposition.
  In logic, [if a^2 is even then a is even] is equivalent to [if a is not even then a^2 is not even].
  If a is not even then it is odd, so there exists a natural number q such that $a = 2 \times q + 1$.
  So $a^2 = (2 \times q + 1)^2 = (2q + 1)(2q + 1) = 4q^2 + 4q + 1 = 2(2q^2 + 2q) + 1$ so a^2 is odd.
  Therefore a^2 is not even.

Application : Find all the divisors of a number

Example : Find all the divisors of 60.

$$\begin{aligned}60 &= 1 \times 60\\60 &= 2 \times 30\\60 &= 3 \times 20\\60 &= 4 \times 15\\60 &= 5 \times 12\\60 &= 6 \times 10\end{aligned}$$

The list of divisors of 60 is thus : $\{1\,;\,2\,;\,3\,;\,4\,;\,5\,;\,6\,;\,10\,;\,12\,;\,15\,;\,20\,;\,30\,;\,60\}$.

Application : Decompose a number into prime factors

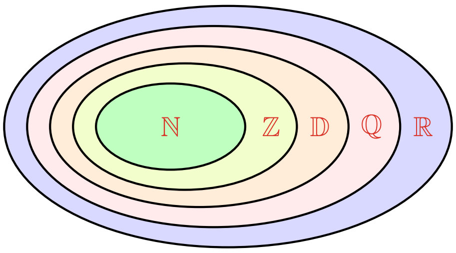
Example : Decompose the number 60 into a product of prime factors.

$$\begin{array}{r|l}60 & 2\\30 & 2\\15 & 3\\5 & 5\\1 & \end{array}$$

We therefore have : $60 = 2^2 \times 3 \times 5$.

I - Set of real numbers

Definition 1 : We call $\mathbb{R}$ **the set of real numbers**. $\mathbb{R} = \{x, x^2 \geq 0\}$



$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{D} \subset \mathbb{Q} \subset \mathbb{R}$$

Remarque : Real numbers that are not rational are called irrational; $\sqrt{2}$ and π are irrational.

Démonstration : Prove that $\sqrt{2} \notin \mathbb{Q}$.

Proof by contradiction.

If $\sqrt{2} \in \mathbb{Q}$ then $\sqrt{2} = \dfrac{n}{d}$ with $n \in \mathbb{N}, d \in \mathbb{N}, d \neq 0$. Moreover, we can choose n and d coprime so that the fraction $\dfrac{n}{d}$ is irreducible.

If $\sqrt{2} = \dfrac{n}{d}$ then $\left(\sqrt{2}\right)^2 = \left(\dfrac{n}{d}\right)^2$ so $2 = \dfrac{n^2}{d^2}$ so $n^2 = 2 \times d^2$, thus n^2 is even which means n is even.

Therefore, there exists a natural number q such that $n = 2 \times q$ yielding $n^2 = (2 \times q)^2 = 4 \times q^2$.

But $n^2 = 2 \times d^2$, so $2 \times d^2 = 4 \times q^2$, thus $d^2 = 2 \times q^2$, therefore d^2 is even which means d is even.

Therefore, there exists a non-zero natural number q' such that $d = 2 \times q'$.

So $\dfrac{n}{d} = \dfrac{2 \times q}{2 \times q'}$ is not irreducible, which contradicts the initial assumption.

Therefore we cannot find 2 natural numbers n and d such that $\sqrt{2} = \dfrac{n}{d}$ with $n \in \mathbb{Z}, d \in \mathbb{N}, d \neq 0$.

Thus $\sqrt{2} \notin \mathbb{Q}$.

Property 1 : Every real number is associated with the abscissa (coordinate) of a point on the number line, and vice-versa.

Méthode : Bound a real number between two decimal numbers

- Bound $\sqrt{2}$ between two decimal numbers to the nearest 10^{-3}.

  1 - First, evaluate $\sqrt{2}$ on a calculator; we obtain $\sqrt{2} \approx 1.4142414$.

  2 - To bound it to the nearest 10^{-3}, we must bound $\sqrt{2}$ between two decimals with 3 decimal places whose difference is less than or equal to 10^{-3}.

  We truncate the obtained value to 3 decimal places; we obtain $\sqrt{2} \geq 1.414$.

  We add 10^{-3} which is 0.001 and we obtain $\sqrt{2} \leq 1.415$.

  3 - We give the desired bounds : $1.414 \leq \sqrt{2} \leq 1.415$.

- Bound $-\sqrt{3}$ between two decimal numbers to the nearest 10^{-2}.

  1 - First, evaluate $-\sqrt{3}$ on a calculator; we obtain $-\sqrt{3} \approx -1.732051$.

  2 - To bound it to the nearest 10^{-2}, we must bound $-\sqrt{3}$ between two decimals with 2 decimal places whose difference is less than or equal to 10^{-2}.

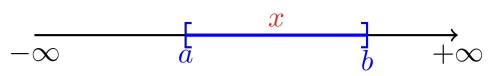
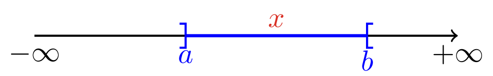
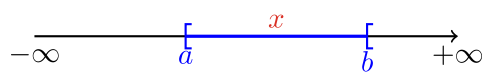
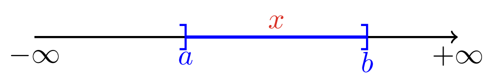
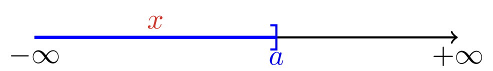
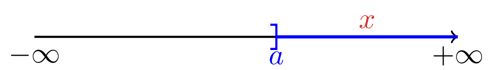
  We truncate the obtained value to 2 decimal places; we obtain $-\sqrt{3} \leq -1.73$.

  We subtract 10^{-2} which is 0.01 and we obtain $-\sqrt{3} \geq -1.74$.

  3 - We give the desired bounds : $-1.74 \leq -\sqrt{3} \leq -1.73$.

II - Intervals

An interval is a set of numbers limited by 2 bounds, which are numbers, $+\infty$ or $-\infty$.
We study the different cases in the table below ($a \leq b$).

Interval	**Inequalities**	**Number line**
$x \in [a;b]$	$a \leq x \leq b$	
$x \in]a;b[$	$a < x < b$	
$x \in [a;b[$	$a \leq x < b$	
$x \in]a;b]$	$a < x \leq b$	
$x \in]-\infty;a]$	$x \leq a$	
$x \in]a;+\infty[$	$x > a$	

Notation : A set of numbers can be composed of the union of several intervals.
In this case, we use the symbol $\cup$ which means or.
Examples :
- $x \in [-1;2] \cup]3;4]$ means that $-1 \leq x \leq 2$ or $3 < x \leq 4$.
- $x \in]-\infty;0[\cup]0;+\infty[$ means that $x < 0$ or $x > 0$.

A set of numbers can be composed of the intersection of several intervals.
In this case, we use the symbol $\cap$ which means and.
Example : $x \in [-1;3] \cap]2;4]$ means that $-1 \leq x \leq 3$ and $2 < x \leq 4$ thus $x \in]2;3]$.

SQUARE ROOT AND ABSOLUTE VALUE

I - Square roots

Definition 1 : The square root of a number a greater than or equal to zero, is the positive number denoted $\sqrt{a}$ such that :

$$(\sqrt{a})^2 = a \quad (a \geq 0)$$

Property 1 : For all positive or zero numbers a and b :

- $\sqrt{a} \times \sqrt{b} = \sqrt{a \times b}$
- $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}} \quad (b \neq 0)$
- $\sqrt{a^2} = a \quad (a \geq 0)$

Application : Simplify a square root

- $\sqrt{48} = \sqrt{16 \times 3} = \sqrt{16} \times \sqrt{3} = \sqrt{4^2} \times \sqrt{3} = 4\sqrt{3}$
- $\sqrt{\dfrac{7}{25}} = \dfrac{\sqrt{7}}{\sqrt{25}} = \dfrac{\sqrt{7}}{\sqrt{5^2}} = \dfrac{\sqrt{7}}{5}$

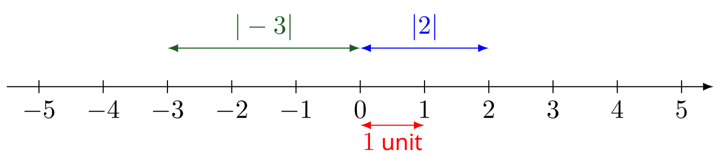
II - Absolute value

Definition 2 : We call the **absolute value** of a real number x the distance between 0 and x on the number line. It is denoted $|x|$.

Property 2 : For any real number x :

$$\begin{cases} |x| = x & \text{if } x \geq 0 \\ |x| = -x & \text{if } x < 0 \end{cases}$$

Explanatory example :



- The distance between 0 and 2 is equal to 2 units, thus $|2| = 2$.
  So if $x \geq 0$ then $|x| = x$.

- The distance between 0 and -3 is equal to 3 units, thus $|-3| = 3$.
  But the opposite of -3 is denoted $-(-3)$ and is equal to 3. Therefore $|-3| = 3 = -(-3)$.
  So if $x < 0$ then $|x| = -x$.

Property 3 : For any real number x, $\sqrt{x^2} = |x|$

Explanatory example :

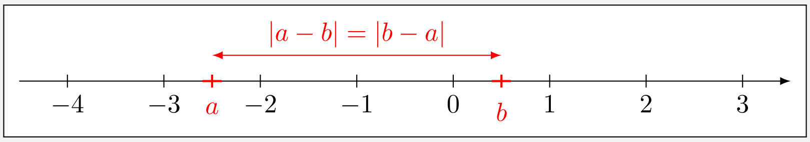
- $\sqrt{2^2} = \sqrt{4}$. By definition, $(\sqrt{4})^2 = 4 = 2^2$ therefore, as $\sqrt{4} \geq 0$ and $2 \geq 0$, we deduce that $\sqrt{4} = 2$.
  Therefore $\sqrt{2^2} = 2 = |2|$.

- $\sqrt{(-2)^2} = \sqrt{4}$. By definition, $(\sqrt{4})^2 = 4 = 2^2$ therefore, as $\sqrt{4} \geq 0$ and $2 \geq 0$, we deduce that $\sqrt{4} = 2$.
  Therefore $\sqrt{(-2)^2} = 2 = |-2|$.

Démonstration :

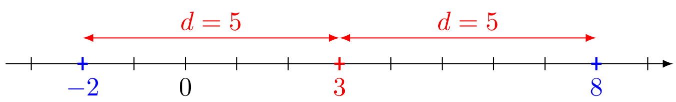
- If $x \geq 0$ then $\sqrt{x^2} = \sqrt{x \times x} = \sqrt{x} \times \sqrt{x} = (\sqrt{x})^2 = x = |x|$.
- If $x < 0$ then $\sqrt{x^2} = \sqrt{(-x)^2} = \sqrt{(-x) \times (-x)} = (\sqrt{-x})^2 = -x = |x|$.

Property 4 : The distance between 2 numbers a and b is equal to $|a - b|$.



Application : Solve an equation with absolute value $|x - 3| = 5$

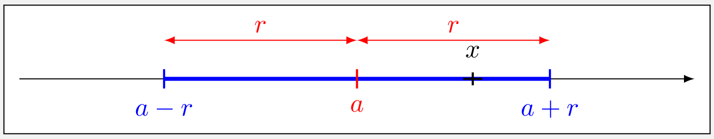
According to the previous property, the equation $|x - 3| = 5$ means that the **distance** between the numbers x and 3 is equal to 5.
On the number line, we therefore look for points located at a distance of 5 from the coordinate 3.



We find two solutions graphically : $3 - 5 = -2$ and $3 + 5 = 8$.
The set of solutions of the equation is therefore $S = \{-2\ ;\ 8\}$.

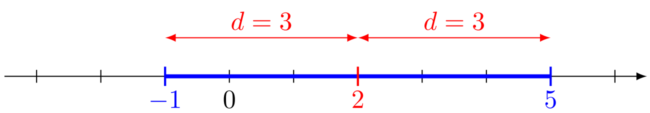
Property 5 : Let a and r be real numbers, $r > 0$;

$$x \in [a - r\ ;\ a + r] \iff |x - a| \leq r$$
$$x \in]a - r\ ;\ a + r[\iff |x - a| < r$$



Application : Solve an inequality with absolute value $|x - 2| \leq 3$

This inequality translates to : the distance between x and 2 is less than or equal to 3.



According to property 5, this corresponds to the interval $[2 - 3\ ;\ 2 + 3]$.
The set of solutions of the inequality is therefore $S = [-1\ ;\ 5]$.

LINEAR EQUATIONS IN ONE UNKNOWN

An equation in one unknown is an equality that contains an unknown number, represented by a letter (x, t, y, $\ldots$).

I - Testing a possible solution

Solving an equation in a set I means finding all the values of the unknown contained in I that satisfy the equality.

Méthode : Test if a number is a solution to an equation

Example 1 : Is the number -1 a solution to the equation $5x-3=7x+2$?

If $x=-1$ then $\begin{cases} 5x-3=5\times(-1)-3=-8 \\ 7x+2=7\times(-1)+2=-5 \end{cases}$ therefore if $x=-1$ then $5x-3\neq 7x+2$

Thus -1 is not a solution to the equation.

Example 2 : Is the number $-\frac{5}{2}$ a solution to the equation $5x-3=7x+2$?

If $x=-\frac{5}{2}$ then $\begin{cases} 5x-3=5\times\left(-\frac{5}{2}\right)-3=-\frac{25}{2}-\frac{6}{2}=-\frac{31}{2} \\ 7x+2=7\times\left(-\frac{5}{2}\right)+2=-\frac{35}{2}+\frac{4}{2}=-\frac{31}{2} \end{cases}$ therefore if $x=-\frac{5}{2}$ then $5x-3=7x+2$

Thus $-\frac{5}{2}$ is a solution to the equation.

II - Addition rule

Property 1 : An equality remains true when the same number is added to both sides of the equality.

For all real numbers a, b, c, if $a=b$ then $a+c=b+c$

Application : Group numbers on the same side of an equality.

Example 1 :

$$x-1=1$$
$$x+(-1)=1$$
$$x+\cancel{(-1)}+\cancel{1}=1+1$$
$$x=2$$

Example 2 :

$$3x=2x+1$$
$$3x+(-2x)=\cancel{2x}+\cancel{(-2x)}+1$$
$$1x=1$$
$$x=1$$

III - Multiplication rule

Property 2 : An equality remains true when both sides of the equality are multiplied by the same number.

For all real numbers a, b, c, if $a=b$ then $a\times c=b\times c$

Application : Express the value of x.

Example 1 :

$$3x=5$$
$$3\times x=5$$
$$\frac{1}{3}\times\cancel{3}\times x=\frac{1}{3}\times 5$$
$$x=\frac{5}{3}$$

Example 2 :

$$-\frac{1}{4}x=2$$
$$-\frac{1}{4}\times x=2$$
$$\cancel{-4}\times\left(\cancel{-\frac{1}{4}}\right)\times x=-4\times 2$$
$$x=-8$$

Application : Simplify solving an equation with fractions.

Example : Find an equivalent equation to $\frac{1}{3}x-2=-\frac{2}{5}x+\frac{1}{5}$ without fractions.

$$\frac{1}{3}x-2=-\frac{2}{5}x+\frac{1}{5}$$

$$\frac{1\times 5}{3\times 5}x-\frac{2\times 15}{1\times 15}=-\frac{2\times 3}{5\times 3}x+\frac{1\times 3}{5\times 3}$$

$$\frac{5x-30}{15}=\frac{-6x+3}{15}$$

$$\cancel{15}\times\frac{5x-30}{\cancel{15}}=\cancel{15}\times\frac{-6x+3}{\cancel{15}}$$

$$5x-30=-6x+3$$

IV - Solving a linear equation in one unknown

Méthode : Solving an equation in $\mathbb{R}$

$$2x+\frac{7}{3}=x\frac{\sqrt{3}}{2}-1$$

We reduce it to an equation without fractions.

$$\frac{2x\times 6}{1\times 6}+\frac{7\times 2}{3\times 2}=x\frac{\sqrt{3}\times 3}{2\times 3}-\frac{1\times 6}{1\times 6}$$

$$\cancel{6}\times\frac{12x+14}{\cancel{6}}=\cancel{6}\times\frac{3\sqrt{3}x-6}{\cancel{6}}$$

$$12x+14=3\sqrt{3}x-6$$

We group the "x" terms on one side and the numbers on the other.

$$12x+(-3\sqrt{3}x)+\cancel{14}+(-14)=\cancel{3\sqrt{3}x}+(-3\sqrt{3}x)-6+(-14)$$

$$12x-3\sqrt{3}x=-20$$

We simplify the expression or factor out "x" then we deduce the expression for "x".

$$(12-3\sqrt{3})x=-20 \quad (12-3\sqrt{3}\neq 0 \text{ so it has a reciprocal})$$

$$\frac{1}{12-3\sqrt{3}}\times\cancel{(12-3\sqrt{3})}x=\frac{1}{12-3\sqrt{3}}\times(-20) \quad \text{thus} \quad x=\frac{-20}{12-3\sqrt{3}}$$

We state the set of solutions to the equation in $\mathbb{R}$

$$S=\left\{\frac{-20}{12-3\sqrt{3}}\right\}$$

Property 3 : A linear equation in one unknown has 0 solutions or 1 solution or else, every real number is a solution to the equation.

Méthode : The different situations to distinguish

Example 1 : The equation $3x=2x+1$ has 1 as its unique solution in $\mathbb{R}$. We denote $S=\{1\}$.
Example 2 : The equation $3x=3x+1$ has no solution in $\mathbb{R}$. We denote $S=\emptyset$.
Example 3 : The equation $3x+1=3x+1$ admits any real number as a solution in $\mathbb{R}$. We denote $S=\mathbb{R}$.

Démonstration :

A linear equation in one unknown x is of the form $ax+b=0$ with $a,b\in\mathbb{R}$.

- If $a=0$ then the equation is written as $b=0$, so it is independent of x.
  - If $b=0$ then the equation is written as $0=0$; this equality is always true, therefore $S=\mathbb{R}$.
  - If $b\neq 0$ then the equality $b=0$ is always false, therefore $S=\emptyset$.
- If $a\neq 0$ then the equation is written as $ax=-b$ then $x=\dfrac{-b}{a}$, thus $\dfrac{-b}{a}$ is the solution to the equation.

Uniqueness : Suppose the equation has 2 solutions x_1 and x_2, then $ax_1+b=0$ and $ax_2+b=0$.

Hence $ax_1+b=ax_2+b \iff ax_1=ax_2 \iff a(x_1-x_2)=0$ but $a\neq 0$ so $x_1-x_2=0$, that is $x_1=x_2$

Thus the equation has a unique solution.

LINEAR INEQUALITIES IN ONE UNKNOWN

A linear inequality in one unknown is an inequality that contains an unknown number, represented by a letter (x, t, y, $\dots$).

I - Testing a possible solution

Solving an inequality in a set I means finding all the values of the unknown contained in I that satisfy the inequality.

Méthode : Test if a number is a solution to an inequality

Example 1 : Is the number -2 a solution to the inequality $5x - 3 > 7x + 2$?

If $x = -2$ then $\begin{cases} 5x - 3 = 5 \times (-2) - 3 = -13 \\ 7x + 2 = 7 \times (-2) + 2 = -12 \end{cases}$ therefore if $x = -2$ then $5x - 3 < 7x + 2$

Thus -2 is not a solution to the inequality.

Example 2 : Is the number -3 a solution to the inequality $5x - 3 > 7x + 2$?

If $x = -3$ then $\begin{cases} 5x - 3 = 5 \times (-3) - 3 = -15 - 3 = -18 \\ 7x + 2 = 7 \times (-3) + 2 = -21 + 2 = -19 \end{cases}$ therefore if $x = -3$ then $5x - 3 > 7x + 2$

Thus -3 is a solution to the inequality.

II - Addition rule

Remarque :

- The addition rule applying to equalities and strict inequalities also applies to non-strict inequalities.
- The addition rule for inequalities is used to group numbers on the same side of an inequality.

Property 4 : An inequality remains true when the same number is added to both sides of the inequality.

For all real numbers a, b, c, if $a < b$ then $a + c < b + c$

III - Multiplication rule

Property 5 :

- An inequality remains true when both sides of the inequality are multiplied by the same **strictly positive** number.

  For all real numbers a, b, c, if $a < b$ and $c > 0$ then $a \times c < b \times c$

- An inequality changes direction when both sides of the inequality are multiplied by the same **strictly negative** number.

  For all real numbers a, b, c, if $a < b$ and $c < 0$ then $a \times c > b \times c$

Application : Express the value of x.

Example 1 :

$$3x < 5$$

$$\frac{1}{3} \times \cancel{3} \times x < \frac{1}{3} \times 5 \quad \text{since } \frac{1}{3} > 0$$

$$x < \frac{5}{3}$$

Example 2 :

$$-\frac{1}{4}x \leq 2$$

$$\cancel{-4} \times \left(\cancel{-\frac{1}{4}}\right) \times x \geq -4 \times 2 \quad \text{since } -4 < 0$$

$$x \geq -8$$

IV - Solving a linear inequality in one unknown

Méthode : Solving an inequality in $\mathbb{R}$

The method for solving a linear inequality in one unknown is identical to the method for solving a linear equation in one unknown.

$$2x+1 \geq \pi x-1$$

We group the "x" terms on one side and the numbers on the other, then simplify or factor out "x".

$$2x+(-\pi x)+\cancel{1}+(-1) \geq \cancel{\pi x}+(-\pi x)-1+(-1)$$

$$2x-\pi x \geq -2$$

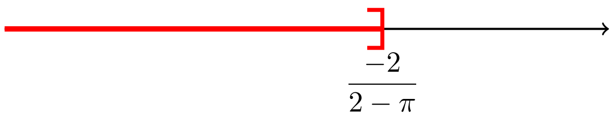
$$(2-\pi)x \geq -2$$

We deduce the expression for "x".

$(2-\pi<0$ so we change the direction of the inequality$)$

$$\frac{1}{2-\pi}\times\cancel{(2-\pi)}x \leq \frac{1}{2-\pi}\times(-2)$$

$$x \leq \frac{-2}{2-\pi}$$



We give the set of solutions to the inequality in $\mathbb{R}$ as an interval :

$$S=\left]-\infty\,;\,\frac{-2}{2-\pi}\right]$$

SYSTEMS OF TWO EQUATIONS IN TWO UNKNOWNS

I - Solving method by elimination (linear combination)

We multiply both sides of each equation in the system by a number such that when we add the 2 equations of the system side by side, we eliminate one of the two unknowns.

Méthode : Solve a system by elimination

$$\text{Solve the system } \begin{cases} 2x - 5y = 12 & (E_1) \\ 3x + 4y = -5 & (E_2) \end{cases}$$

We choose, for example, to eliminate the unknown y :
We multiply (E_1) by 4, and we multiply (E_2) by 5. We obtain :

$$\begin{cases} 4 \times 2x - 4 \times 5y = 4 \times 12 \\ 5 \times 3x + 5 \times 4y = -5 \times 5 \end{cases} \quad \text{hence} \quad \begin{cases} 8x - 20y = 48 \\ 15x + 20y = -25 \end{cases}$$

We add the 2 equations side by side. We solve the resulting equation. We obtain :

$$8x + 15x \cancel{-20y + 20y} = 48 - 25 \quad \text{thus} \quad 23x = 23$$

$$x = \frac{23}{23} = 1$$

We replace x by its value in (E_1) or (E_2). With (E_1), we obtain :

$$2 \times 1 - 5y = 12 \quad \text{thus} \quad -5y = 10$$

$$y = \frac{10}{-5} = -2$$

$$\text{thus} \quad S = \{(1\,;-2)\}$$

II - Solving method by substitution

We express one unknown in terms of the other in one of the 2 equations of the system, then we replace this unknown by this value in the other equation.

This method is preferably used when one of the 2 unknowns has a coefficient of 1 or -1.

Méthode : Solve a system by substitution

$$\text{Solve the system } \begin{cases} x + 3y = 2 & (E_1) \\ 2x - y = -1 & (E_2) \end{cases}$$

We choose, for example, to express x in terms of y in (E_1) then we replace x in (E_2) by this value. We obtain :

$$\begin{cases} x = 2 - 3y \\ 2(2 - 3y) - y = -1 \end{cases}$$

We solve the resulting equation with unknown y. We obtain :

$$2 \times 2 - 2 \times 3y - y = -1 \quad \text{thus} \quad 4 - 7y = -1 \quad \text{thus} \quad -7y = -5$$

$$y = \frac{-5}{-7} = \frac{5}{7}$$

We replace y by its value in (E_1). We obtain :

$$x + 3 \times \frac{5}{7} = 2$$

$$x = 2 - \frac{15}{7}$$

$$x = \frac{14}{7} - \frac{15}{7} = \frac{-1}{7}$$

$$\text{thus} \quad S = \left\{\left(-\frac{1}{7}\,;\frac{5}{7}\right)\right\}$$

QUADRATIC EQUATIONS IN ONE UNKNOWN

I - Algebraic resolution of a product equation

Property 1 : If a product of factors is zero then at least one of its factors is zero.

$$a, b \in \mathbb{R}, \text{ if } a \times b = 0 \text{ then } a = 0 \text{ or } b = 0$$

Application : Solve an equation of the type $(ax + b)(cx + d) = 0$.

Resolution of the equation $(x + 5)(3x - 2) = 0$

If $(x + 5)(3x - 2) = 0$ then $x + 5 = 0$ or $3x - 2 = 0$

$$\text{that is} \quad x = -5 \quad \text{or} \quad x = \frac{2}{3}$$

$$\text{Thus } S = \left\{-5\,;\,\frac{2}{3}\right\}$$

II - Algebraic resolution of a quadratic equation

A quadratic equation in one unknown is an equation of the form $ax^2 + bx + c = 0$ with $a, b, c \in \mathbb{R}$ and $a \neq 0$.
Indeed, if $a = 0$, we obtain a linear equation in one unknown.

When solving such an equation, the goal is to reduce it to a product equation in order to use property 1 to determine the set of solutions.

1 - Factorize using a notable identity

Property 2 : Let a, b, c be real numbers;

Expanded form   Factored form

$$ab + ac = a(b + c)$$
$$a^2 + 2ab + b^2 = (a + b)^2$$
$$a^2 - 2ab + b^2 = (a - b)^2$$
$$a^2 - b^2 = (a + b)(a - b)$$

Méthode : Resolution of a quadratic equation

Solve the equation $(x - 1)^2 = 2(x - 1)$ in $\mathbb{R}$

1 - We group all the terms on one side

$$(x - 1)^2 - 2(x - 1) = 0$$

2 - We factorize using a notable identity

$$\begin{aligned}(x - 1) \times (x - 1) - 2 \times (x - 1) &= 0\\ (x - 1) \times [(x - 1) - 2] &= 0\\ (x - 1) \times (x - 1 - 2) &= 0\\ (x - 1)(x - 3) &= 0\end{aligned}$$

3 - We apply property 1

$$x - 1 = 0 \quad \text{or} \quad x - 3 = 0$$
$$x = 1 \quad \text{or} \quad x = 3$$

4 - We give the set of solutions

$$S = \{1\,;\,3\}$$

Méthode : Solve an equation of the type $x^2 = a$, a being a given real number.

- If $a < 0$ then $S = \varnothing$ *(the square of a number is always positive or zero)*.
- If $a = 0$ then $S = \{0\}$.
- If $a > 0$ then we use the general method studied previously :

$$\begin{aligned} x^2 &= a \\ x^2 - a &= 0 \\ x^2 - \left(\sqrt{a}\right)^2 &= 0 \\ \left(x + \sqrt{a}\right)\left(x - \sqrt{a}\right) &= 0 \end{aligned}$$

$$x + \sqrt{a} = 0 \quad \text{or} \quad x - \sqrt{a} = 0$$

$$x = -\sqrt{a} \quad \text{or} \quad x = \sqrt{a}$$

$$S = \left\{-\sqrt{a}\,;\, \sqrt{a}\right\}$$

Programme en Python : solve an equation of the type $x^2 = a$

```
from math import *

print("Solve an equation of the type x*x = a")
a = float(input("enter a = "))

if a < 0:
   print("No solution")
elif a == 0:
   print("S = {0}")
else:
   x1 = -sqrt(a)
   x2 = sqrt(a)
   print("S = {", x1, ";", x2, "}")
```

QUOTIENT EQUATIONS IN ONE UNKNOWN

A quotient equation is an equation containing fractions where the unknown is present in the denominator.
What changes in this case is that certain values of the unknown that make the denominator of a fraction in the equation zero must be excluded from the set of solutions. These values are called forbidden values.

Property 1 : If a and b are real numbers such that $b \neq 0$ then :

$$\frac{a}{b} = 0 \iff a = 0$$

Méthode : Solving a quotient equation

Resolution of the equation $\dfrac{3x+1}{1-x} = \dfrac{1}{x+1}$

1 - Determine the forbidden values

$$1 - x = 0 \quad \text{and} \quad x + 1 = 0$$

$$x = 1 \quad \text{and} \quad x = -1$$

2 - Group all the terms on one side

$$\frac{3x+1}{1-x} - \frac{1}{x+1} = 0$$

3 - Put all fractions over a common denominator

$$\frac{(3x+1)(x+1)}{(1-x)(x+1)} - \frac{1(1-x)}{(x+1)(1-x)} = 0$$

$$\frac{(3x+1)(x+1) - 1(1-x)}{(x+1)(1-x)} = 0$$

4 - Factor the numerator using a notable identity
(or expand then simplify if no factorization is possible)

$$\frac{3x^2 + x + 3x + 1 - 1 + x}{(x+1)(1-x)} = 0$$

$$\frac{3x^2 + 5x}{(x+1)(1-x)} = 0$$

$$\frac{x(3x+5)}{(x+1)(1-x)} = 0$$

5 - Calculate the roots of the numerator

$$x = 0 \quad \text{or} \quad 3x + 5 = 0$$

$$x = -\frac{5}{3}$$

6 - Eliminate any forbidden values from the roots obtained in step 5

0 and $-\dfrac{5}{3}$ are not forbidden values.

7 - State the set of solutions

$$S = \left\{0\,;\,-\frac{5}{3}\right\}$$

QUADRATIC INEQUALITIES IN ONE UNKNOWN

I - Algebraic resolution of a product inequality

Property 1 :

x	$-\infty$		$-\frac{b}{a}$		$+\infty$
Sign of $ax+b$		sign of $-a$	0	sign of a	

Property 2 :

- The product of 2 real numbers with the same sign is positive.
- The product of 2 real numbers with opposite signs is negative.

Méthode : Resolution of an inequality of the type $(ax+b)(cx+d) \leq 0$.

Solve the inequality $(x-2)(-3x-6) \leq 0$ in $\mathbb{R}$

1 - Finding the roots of the polynomial

$$x-2=0 \quad \text{or} \quad -3x-6=0$$
$$x=2 \quad \text{or} \quad x=-2$$

2 - Sign table

x		$-\infty$		-2		2		$+\infty$
$a=1>0$	$x-2$		$-$		$-$	0	$+$	
$a=-3<0$	$-3x-6$		$+$	0	$-$		$-$	
$(x-2)(-3x-6)$			$-$	0	$+$	0	$-$	

3 - Set of solutions :

$$S =]-\infty\,;\,-2] \cup [2\,;\,+\infty[$$

II - Algebraic resolution of a quadratic inequality

The beginning of the resolution is identical to that of solving a quadratic equation in one unknown.
The goal is to obtain a product inequality in order to then use the method studied in paragraph I.

Méthode : Resolution of a quadratic inequality

Solve the inequality $(x-1)^2 > 2(x-1)$ in $\mathbb{R}$

1 - We group all the terms on one side

$$(x-1)^2 - 2(x-1) > 0$$

2 - We factorize using a notable identity

$$\begin{aligned}(x-1)\times(x-1) - 2\times(x-1) &> 0\\ (x-1)\times[(x-1)-2] &> 0\\ (x-1)\times(x-1-2) &> 0\\ (x-1)(x-3) &> 0\end{aligned}$$

3 - We find the roots of the polynomial

$$x-1=0 \quad \text{or} \quad x-3=0$$

$$x=1 \quad \text{or} \quad x=3$$

4 - We make a sign table

x	$-\infty$		1		3		$+\infty$
$a=1>0$ $\quad x-1$		$-$	0	$+$		$+$	
$a=1>0$ $\quad x-3$		$-$		$-$	0	$+$	
$(x-1)(x-3)$		$+$	0	$-$	0	$+$	

5 - We state the set of solutions :

$$S =]-\infty\,;\,1[\,\cup\,]3\,;\,+\infty[$$

Property 1 :

- The quotient of 2 real numbers with the same sign is positive.
- The quotient of 2 real numbers with opposite signs is negative.

Méthode : Solving a quotient inequality

Solve the inequality $\dfrac{x+1}{x+2} < \dfrac{x-1}{x-2}$ in $\mathbb{R}$

1 - Determine the forbidden values

$$x+2=0 \quad \text{and} \quad x-2=0$$
$$x=-2 \quad \text{and} \quad x=2$$

2 - Group all the terms on one side

$$\frac{x+1}{x+2} - \frac{x-1}{x-2} < 0$$

3 - Put all fractions over a common denominator

$$\frac{(x+1)(x-2)}{(x+2)(x-2)} - \frac{(x-1)(x+2)}{(x-2)(x+2)} < 0$$

$$\frac{(x+1)(x-2)-(x-1)(x+2)}{(x+2)(x-2)} < 0$$

4 - Factor the numerator using a notable identity
(or expand if no factorization is possible)

$$\frac{x^2+x-2x-2-(x^2-x+2x-2)}{(x+2)(x-2)} < 0$$

$$\frac{\cancel{x^2}+x-2x\cancel{-2}-\cancel{x^2}+x-2x+\cancel{2}}{(x+2)(x-2)} < 0$$

$$\frac{-2x}{(x+2)(x-2)} < 0$$

5 - Calculate the roots of the numerator

$$-2x=0$$
$$x=0$$

6 - Make a sign table

	x	$-\infty$		-2		0		2		$+\infty$
$a=1>0$	$x+2$		$-$	0	$+$		$+$		$+$	
$a=1>0$	$x-2$		$-$		$-$		$-$	0	$+$	
$a=-2<0$	$-2x$		$+$		$+$	0	$-$		$-$	
	$\dfrac{-2x}{(x+2)(x-2)}$		$+$	$\|\|$	$-$	0	$+$	$\|\|$	$-$	

7 - State the set of solutions :

$$S =]-2\,;\,0[\cup]2\,;\,+\infty[$$

GENERALITIES ON FUNCTIONS

In middle school, a function f is a mathematical relation that associates a number x with the number $f(x)$.
We denote it $f : x \mapsto f(x)$ which is read "the function f which associates $f(x)$ to x".
The number x is called the **variable**. Indeed, it is by varying this number that we will calculate the different values of $f(x)$.

I - Domain of definition

Definition 1 : We call a **real function** f a relation between 2 sets of real numbers D and F such that, to each element of D, we associate a unique element of F. We denote :

$$f : \text{D} \to \text{F}$$
$$x \mapsto f(x)$$

Vocabulaire :

- x is called the **antecedent** of $f(x)$.
- $f(x)$ is called the **image** of x.
- D is called the **domain of definition** of f.

Some reference functions and their domain of definition

- The affine function $x \mapsto ax + b$, $a, b \in \mathbb{R}$ is defined on $\mathbb{R}$ (if $b = 0$, we say that the function is linear).
- The square function $x \mapsto x^2$ is defined on $\mathbb{R}$.
- The reciprocal function $x \mapsto \dfrac{1}{x}$ is defined on $]-\infty; 0[\cup]0; +\infty[$ (the number 0 has no reciprocal).
- The square root function $x \mapsto \sqrt{x}$ is defined on $[0; +\infty[$ (the square root of a negative number does not exist).

Remarque : Depending on the needs, we can restrict the extent of the domain of definition of a function.

II - Representative curve

One advantage of functions is that they can be associated with curves.

Definition 2 : We call the **representative curve of the function** f the set

$$C_f = \{M(x, y), x \in \text{D}, y \in \text{F such that } y = f(x)\}$$

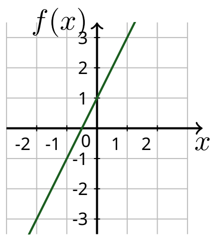
Remarque :

- We also say that the curve C_f has the **equation** $y = f(x)$.
- To sketch the shape of the representative curve of a function on a given interval, we calculate the coordinates of a sufficient number of points on the curve then connect them (without a ruler).

Some reference curves

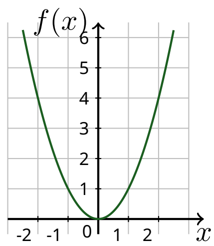
Affine function

$y = 2x + 1$



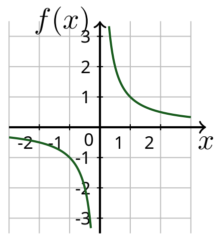
Square function

$y = x^2$



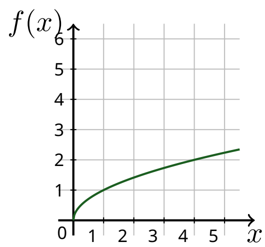
Reciprocal function

$y = \dfrac{1}{x}$



Square root function

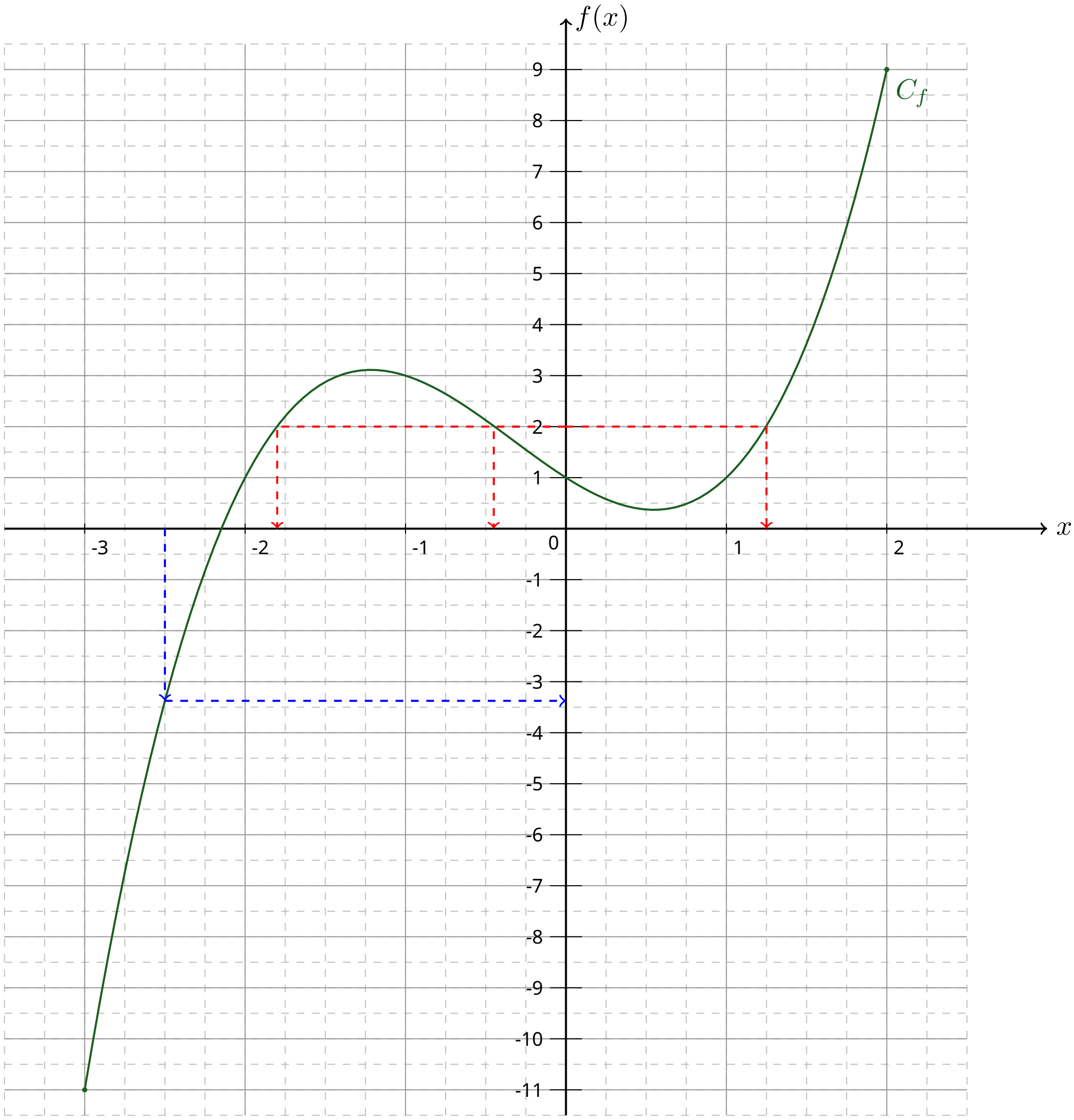
$y = \sqrt{x}$



III - Graphical readings

Graphical reading of image and antecedents

Below, the representative curve of the function $f : x \mapsto x^3 + x^2 - 2x + 1$ defined on D $= [-3; 2]$ has been drawn.



- Graphically read an image
  The ordinate of the point on the curve C_f with abscissa -2.5 is about -3.5.
  Which means that the image of -2.5 by f is about -3.5 therefore $f(-2.5) \approx -3.5$.

- Graphically read antecedents
  Three points on the curve C_f have an ordinate of 2.
  Their respective abscissas are about -1.8; -0.45; 1.25.
  The antecedents of 2 by f are therefore about -1.8; -0.45; 1.25
  thus $f(-1.8) \approx 2$; $f(-0.45) \approx 2$; $f(1.25) \approx 2$.

  No point on the curve C_f has an ordinate equal to -11.2 so -11.2 has no antecedent by f.

Remarque :

- Graphically solving the equation $f(x) = k$ amounts to graphically reading the antecedents of k by the function f.
- Graphically solving the inequality $f(x) > k$ amounts to graphically reading the abscissas of the points on the representative curve of f having an ordinate strictly greater than k.

IV - Study of a function

1 – Direction of variation

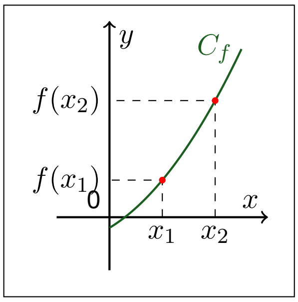
On a graph, when the curve goes up, we say that the function is increasing and when the curve goes down, we say that the function is decreasing.

Graphical reading of the direction of variation

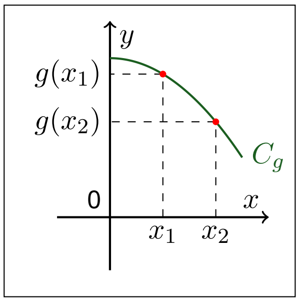
In the previous example, by graphical reading :
f is increasing on $[-3; -1.25] \cup [0.5; 2]$ and decreasing on $[-1.25; 0.5]$.

To study the direction of variation of a function, a mathematical interpretation of these observations is necessary.

Direction of variation of a function



C_f goes up means that the function f is increasing.
Let $(x_1, f(x_1))$ and $(x_2, f(x_2))$ be two points on C_f, we observe that if $x_1 \leq x_2$ then $f(x_1) \leq f(x_2)$.



C_g goes down means that the function g is decreasing.
Let $(x_1, g(x_1))$ and $(x_2, g(x_2))$ be two points on C_g, we observe that if $x_1 \leq x_2$ then $g(x_1) \geq g(x_2)$.

Definition 3 : Let f be a function defined on an **interval** I.

- We say that the **function** f is **increasing on I**, when for all real numbers x_1 and x_2 in I, if $x_1 \leq x_2$ then $f(x_1) \leq f(x_2)$
- We say that the **function** f is **decreasing on I**, when for all real numbers x_1 and x_2 in I, if $x_1 \leq x_2$ then $f(x_1) \geq f(x_2)$

Remarque :

The direction of variation of a function is studied locally, meaning that x_1 and x_2 are very close. The function f must therefore be defined on $[x_1; x_2]$. This is why in the definition, we require the function f to be defined on an interval I containing x_1 and x_2.

2 – Maximum and minimum of a function

Definition 4 : Let f be a function defined on an **interval** I and a a real number belonging to I.

- We say that $f(a)$ is the **maximum** of the function f **on I**, when :
  for any real number x in I, $f(x) \leq f(a)$
- We say that $f(a)$ is the **minimum** of the function f **on I**, when :
  for any real number x in I, $f(x) \geq f(a)$

Graphical reading of extrema

In the previous example, by graphical reading :

- The minimum of f on D is $f(-3) = -11$ and the maximum of f on D is $f(2) = 9$.
- The maximum of f on $[-2; 0]$ is $f(-1.25) \approx 3$.

3 – Variation table of a function

The variation table of a function summarizes the study of a function on its domain of definition.

Variation table

For the previous example, we obtain the following table :

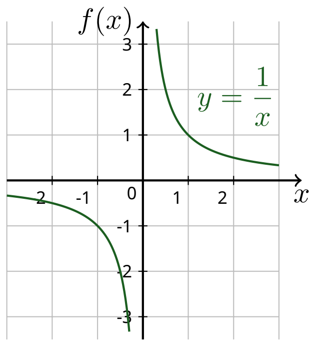
x	-3		-1.25		0.5		2
$f(x)$	-11	$\nearrow$	3	$\searrow$	0.4	$\nearrow$	9

An upward arrow indicates that the function is increasing, and a downward arrow indicates that the function is decreasing.

Variation table and forbidden value

The reciprocal function $f : x \mapsto \dfrac{1}{x}$ is defined on $]-\infty;0[\cup]0;+\infty[$.
We place a double bar in the variation table to indicate that 0 has no image.

x	$-\infty$		0		$+\infty$
$f(x) = \dfrac{1}{x}$		$\searrow$	$\|$	$\searrow$	



I - Definition and graphical representation

Definition 1 : An **affine function** is a function defined on $\mathbb{R}$ by $x \mapsto ax + b$ with $a, b \in \mathbb{R}$.

Remarque : A **linear function** $x \mapsto ax$ defined on $\mathbb{R}$ is a particular affine function.

Property 1 : The graphical representation of the function defined on $\mathbb{R}$ by $x \mapsto ax + b$ with $a, b \in \mathbb{R}$ is the line with equation $y = ax + b$.

Graphical representation of an affine function

Graphical representation of f defined on $\mathbb{R}$ by $x \mapsto 3x - 4$.
The graphical representation of f is the line $D_f : y = 3x - 4$.

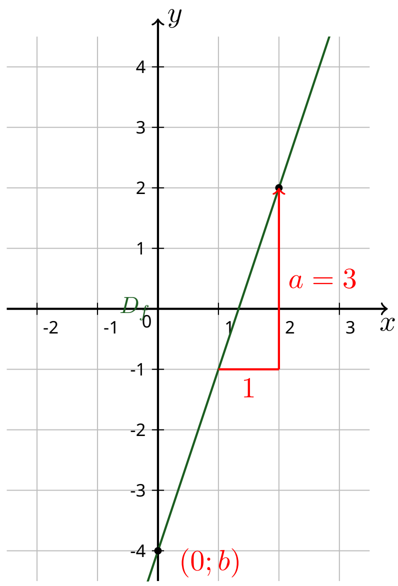
To draw this line, it is sufficient to have the coordinates of 2 distinct points on this line.

$f(0) = 3 \times 0 - 4 = -4$ so $(0; -4) \in D_f$

$f(2) = 3 \times 2 - 4 = 2$ so $(2; 2) \in D_f$.

The **slope** (or gradient) of D_f is the difference between the images of 2 consecutive integers : $a = f(2) - f(1) = 2 - (3 \times 1 - 4) = 3$.
(This explains the graphical method for determining the slope.)

The **y-intercept** of D_f is the ordinate of the intersection point of D_f with the (Oy) axis : $b = -4$



II - Direction of variation and sign ($a \neq 0$)

Direction of variation of $f : x \mapsto ax + b$ with $a \neq 0$

Let x_1 and x_2 be real numbers such that $x_1 < x_2$;

If $a < 0$ then $a \times x_1 > a \times x_2$
hence $a \times x_1 + b > a \times x_2 + b$
so $f(x_1) > f(x_2)$
thus f is decreasing.

x	$-\infty$ $\qquad$ $+\infty$
$f(x) = ax + b$ $a < 0$	$\searrow$

If $a > 0$ then $a \times x_1 < a \times x_2$
hence $a \times x_1 + b < a \times x_2 + b$
so $f(x_1) < f(x_2)$
thus f is increasing.

x	$-\infty$ $\qquad$ $+\infty$
$f(x) = ax + b$ $a > 0$	$\nearrow$

Sign of $f : x \mapsto ax + b$ with $a \neq 0$

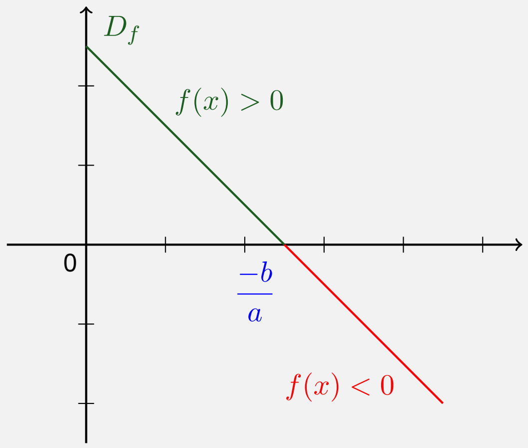
If $a < 0$ then $\quad ax + b > 0$
$\Leftrightarrow \quad ax > -b$
$\Leftrightarrow \quad x < -\dfrac{b}{a}$ (since $a < 0$)

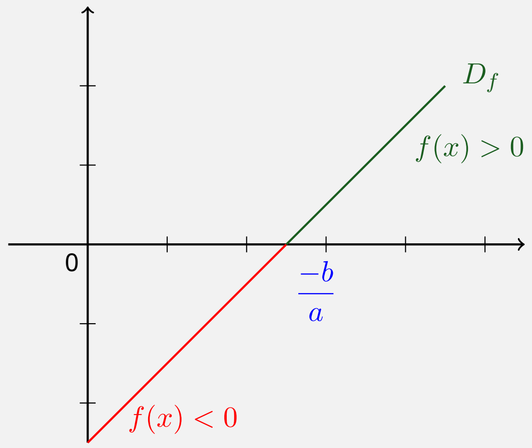
If $a > 0$ then $\quad ax + b > 0$
$\Leftrightarrow \quad ax > -b$
$\Leftrightarrow \quad x > -\dfrac{b}{a}$ (since $a > 0$)

Property 2 : Summary table of an affine function

x	$-\infty$	$\frac{-b}{a}$	$+\infty$
$f(x)=ax+b$ $a<0$	$\searrow$		
Sign of $ax+b$	$+$	0	$-$

x	$-\infty$	$\frac{-b}{a}$	$+\infty$
$f(x)=ax+b$ $a>0$	$\nearrow$		
Sign of $ax+b$	$-$	0	$+$





THE SQUARE FUNCTION

Definition 1 : We call the **square function** the function defined on $\mathbb{R}$ by $x \mapsto x^2$.

Study of the variations of the square function

Let $x_1, x_2 \in]-\infty; 0]$ such that $x_1 \leq x_2$

$$x_2^2 - x_1^2 = (x_2 + x_1)(x_2 - x_1)$$

$x_1 \leq 0$ and $x_2 \leq 0$ so $x_2 + x_1 \leq 0$
and $x_1 \leq x_2$ so $x_2 - x_1 \geq 0$

thus $(x_2 + x_1)(x_2 - x_1) \leq 0$

thus $x_2^2 \leq x_1^2$

so $x \mapsto x^2$ **is decreasing on** $]-\infty; 0]$

Let $x_1, x_2 \in [0; +\infty[$ such that $x_1 \leq x_2$

$$x_2^2 - x_1^2 = (x_2 + x_1)(x_2 - x_1)$$

$x_1 \geq 0$ and $x_2 \geq 0$ so $x_2 + x_1 \geq 0$
and $x_1 \leq x_2$ so $x_2 - x_1 \geq 0$

thus $(x_2 + x_1)(x_2 - x_1) \geq 0$

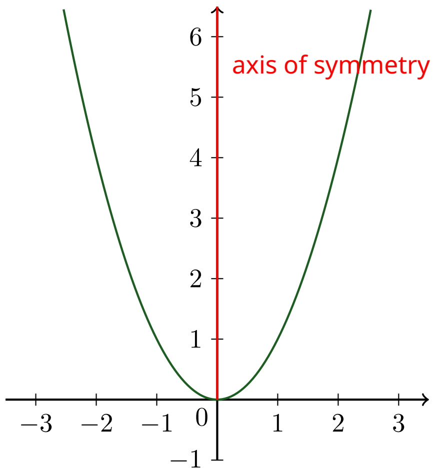
thus $x_2^2 \geq x_1^2$

so $x \mapsto x^2$ **is increasing on** $[0; +\infty[$

We deduce the **variation table of the square function** :

x	$-\infty$	0	$+\infty$
x^2	↘	0	↗

Graphical representation of the square function



The representative curve of the square function is the **parabola** with equation $y = x^2$.

It is above the abscissa axis.
Indeed, **for all** $x \in \mathbb{R}$, $x^2 \geq 0$.

$(0; 0)$ is the only contact point of the parabola with the abscissa axis.
Indeed, $0^2 = 0$.

Application : Comparing expressions

Study the variations of the function $f : x \mapsto -4x^2 + 1$ on $[0; +\infty[$.

$$0 \leq x_1 < x_2$$

$$0 \leq x_1^2 < x_2^2 \qquad \textit{(square function increasing on } [0; +\infty[\textit{)}$$

$$(-4) \times x_1^2 > (-4) \times x_2^2 \qquad \textit{(multiplication by a negative number)}$$

$$-4x_1^2 + 1 > -4x_2^2 + 1$$

$$f(x_1) > f(x_2)$$

Thus the function f is decreasing on $[0; +\infty[$.

THE CUBE FUNCTION

Definition 1 : We call the **cube function** the function defined on $\mathbb{R}$ by $x \mapsto x^3$.

Study of the variations of the cube function

Note : $(a-b)(a^2+ab+b^2) = a^3 + a^2b + ab^2 - a^2b - ab^2 - b^3 = a^3 - b^3$

Let $x_1, x_2 \in]-\infty; 0]$ such that $x_1 \leq x_2$

$x_2^3 - x_1^3 = (x_2 - x_1)(x_2^2 + x_2x_1 + x_1^2)$

$x_1 \leq 0$ and $x_2 \leq 0$ so $x_2x_1 \geq 0$
and $x_1 \leq x_2$ so $x_2 - x_1 \geq 0$

thus $(x_2 - x_1)(x_2^2 + x_2x_1 + x_1^2) \geq 0$

thus $x_2^3 \geq x_1^3$

so $\boldsymbol{x \mapsto x^3}$ **is increasing on** $]-\infty; 0]$

Let $x_1, x_2 \in [0; +\infty[$ such that $x_1 \leq x_2$

$x_2^3 - x_1^3 = (x_2 - x_1)(x_2^2 + x_2x_1 + x_1^2)$

$x_1 \geq 0$ and $x_2 \geq 0$ so $x_2x_1 \geq 0$
and $x_1 \leq x_2$ so $x_2 - x_1 \geq 0$

thus $(x_2 - x_1)(x_2^2 + x_2x_1 + x_1^2) \geq 0$

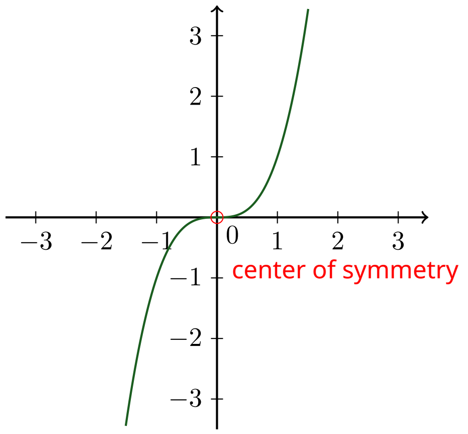
thus $x_2^3 \geq x_1^3$

so $\boldsymbol{x \mapsto x^3}$ **is increasing on** $[0; +\infty[$

We deduce the **variation table of the cube function** :

x	$-\infty$	0	$+\infty$
x^3	↗	0	↗

Graphical representation of the cube function



The representative curve of the cube function is the curve with equation $\boldsymbol{y = x^3}$. It is symmetric with respect to the origin of the coordinate system.

The curve is below the abscissa axis on $]-\infty; 0[$.
Indeed, **for all** $\boldsymbol{x \in]-\infty; 0[, x^3 < 0}$.

The curve is above the abscissa axis on $]0; +\infty[$.
Indeed, **for all** $\boldsymbol{x \in]0; +\infty[, x^3 > 0}$.

$(0; 0)$ is the only contact point of the curve with the abscissa axis.
Indeed, $\boldsymbol{0^3 = 0}$.

Application : Comparing expressions

Study the variations of the function $f : x \mapsto -x^3 + 1$ on $\mathbb{R}$.

$x_1 < x_2$

$x_1^3 < x_2^3$ (*cube function increasing on* $\mathbb{R}$)

$-x_1^3 > -x_2^3$ (*multiplication by a negative number*)

$-x_1^3 + 1 > -x_2^3 + 1$

$f(x_1) > f(x_2)$

Thus the function f is decreasing on $\mathbb{R}$.

THE RECIPROCAL FUNCTION

Definition 1 : We call the **reciprocal function** the function defined on $]-\infty;0[\cup]0;+\infty[$ by $x \mapsto \dfrac{1}{x}$.

Study of the variations of the reciprocal function

Let $x_1, x_2 \in]-\infty;0[$ such that $x_1 \leq x_2$

$$\frac{1}{x_2} - \frac{1}{x_1} = \frac{1 \times x_1}{x_2 \times x_1} - \frac{1 \times x_2}{x_1 \times x_2} = \frac{x_1 - x_2}{x_1 x_2}$$

$x_1 \leq x_2$ so $x_1 - x_2 \leq 0$

and $x_1 < 0$ and $x_2 < 0$ so $x_1 x_2 > 0$

thus $\dfrac{x_1 - x_2}{x_1 x_2} \leq 0$ thus $\dfrac{1}{x_2} \leq \dfrac{1}{x_1}$

so $x \mapsto \dfrac{1}{x}$ **is decreasing on** $]-\infty;0[$

Let $x_1, x_2 \in]0;+\infty[$ such that $x_1 \leq x_2$

$$\frac{1}{x_2} - \frac{1}{x_1} = \frac{1 \times x_1}{x_2 \times x_1} - \frac{1 \times x_2}{x_1 \times x_2} = \frac{x_1 - x_2}{x_1 x_2}$$

$x_1 \leq x_2$ so $x_1 - x_2 \leq 0$

and $x_1 > 0$ and $x_2 > 0$ so $x_1 x_2 > 0$

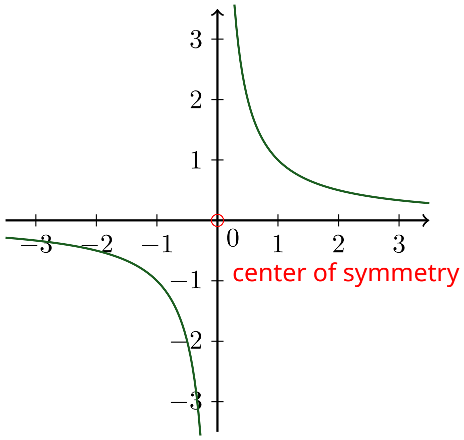
thus $\dfrac{x_1 - x_2}{x_1 x_2} \leq 0$ thus $\dfrac{1}{x_2} \leq \dfrac{1}{x_1}$

so $x \mapsto \dfrac{1}{x}$ **is decreasing on** $]0;+\infty[$

We deduce the **variation table of the reciprocal function** :

x	$-\infty$	0	$+\infty$
$\dfrac{1}{x}$	↘	‖	↘

Graphical representation of the reciprocal function



The representative curve of the reciprocal function is the **hyperbola** with equation $y = \dfrac{1}{x}$.

It is symmetric with respect to the origin of the coordinate system.

It is composed of 2 branches since 0 has no image.

The curve is below the abscissa axis on $]-\infty;0[$.

Indeed, **for all** $x \in]-\infty;0[$, $\dfrac{1}{x} < 0$.

The curve is above the abscissa axis on $]0;+\infty[$.

Indeed, **for all** $x \in]0;+\infty[$, $\dfrac{1}{x} > 0$.

Application : Comparing expressions

Study the variations of the function $f : x \mapsto \dfrac{2}{x} - 1$ on $]0;+\infty[$.

$$0 < x_1 < x_2$$

$$\frac{1}{x_1} > \frac{1}{x_2} > 0 \qquad \textit{(reciprocal function decreasing on }]0;+\infty[\textit{)}$$

$$2 \times \frac{1}{x_1} > 2 \times \frac{1}{x_2} > 0$$

$$\frac{2}{x_1} - 1 > \frac{2}{x_2} - 1$$

$$f(x_1) > f(x_2)$$

Thus the function f is decreasing on $]0;+\infty[$.

THE SQUARE ROOT FUNCTION

Definition 1 : We call the **square root function** the function defined on $[0;+\infty[$ by $x \mapsto \sqrt{x}$.

Study of the variations of the square root function

Let $x_1, x_2 \in [0;+\infty[$ such that $x_1 \leq x_2$

$$\sqrt{x_2}-\sqrt{x_1} = \frac{(\sqrt{x_2}-\sqrt{x_1})(\sqrt{x_2}+\sqrt{x_1})}{\sqrt{x_2}+\sqrt{x_1}} = \frac{(\sqrt{x_2})^2-(\sqrt{x_1})^2}{\sqrt{x_2}+\sqrt{x_1}} = \frac{x_2-x_1}{\sqrt{x_2}+\sqrt{x_1}}$$

$\sqrt{x_2}+\sqrt{x_1} > 0$ and $x_1 \leq x_2$ so $x_2 - x_1 \geq 0$

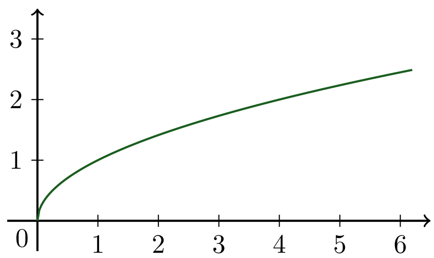
thus $\sqrt{x_2}-\sqrt{x_1} \geq 0$

so $x \mapsto \sqrt{x}$ **is increasing on** $[0;+\infty[$

We deduce the **variation table of the square root function** :

x	0	$+\infty$
$\sqrt{x}$	0 $\nearrow$	

Graphical representation of the square root function



The representative curve of the square root function is the curve with equation $y = \sqrt{x}$.

It is above the abscissa axis.
Indeed, **for all** $x \in [0;+\infty[$, $\sqrt{x} \geq 0$.

$(0;0)$ is the only contact point of the curve with the abscissa axis.
Indeed, $\sqrt{0} = 0$.

Application : Comparing expressions

Study the variations of the function $f : x \mapsto -2\sqrt{x}+3$ on $[0;+\infty[$.

$0 \leq x_1 < x_2$

$\sqrt{x_1} < \sqrt{x_2}$ (*square root function increasing on* $[0;+\infty[$)

$-2\sqrt{x_1} > -2\sqrt{x_2}$ (*multiplication by a negative number*)

$-2\sqrt{x_1}+3 > -2\sqrt{x_2}+3$

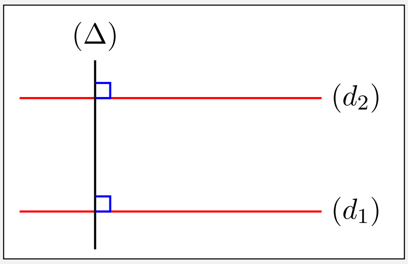
$f(x_1) > f(x_2)$

Thus the function f is decreasing on $[0;+\infty[$.

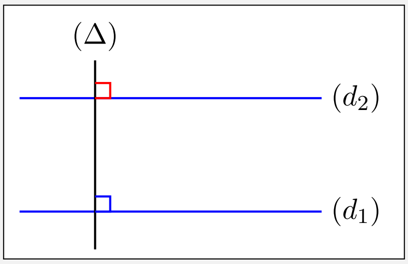
REMINDERS AND COMPLEMENTS IN GEOMETRY

I - Lines

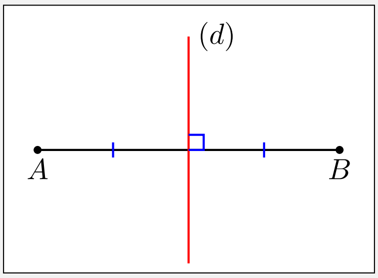
Property 1 : If 2 lines are perpendicular to the same line then they are parallel.



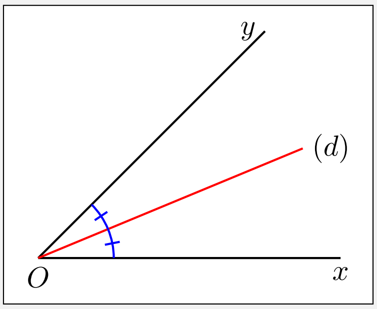
Property 2 : If 2 lines are parallel and a third line is perpendicular to one of them, then it is perpendicular to the other.



Definition 1 : The **perpendicular bisector** of a segment is the line perpendicular to the segment that passes through its midpoint.

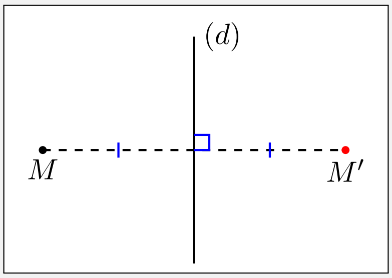


Definition 2 : The **angle bisector** of an angle is a line that divides an angle into 2 equal angles.

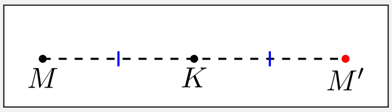


II - Symmetries

Definition 3 : The **symmetric of point M with respect to a line (d)** is the point M' such that (d) is the perpendicular bisector of the segment $[MM']$.

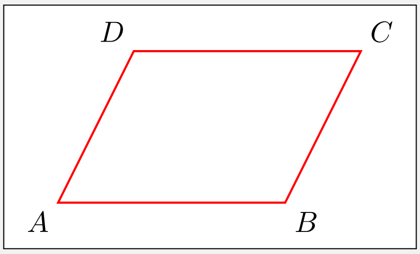


Definition 4 : The **symmetric of point M with respect to a point K** is the point M' such that K is the midpoint of the segment $[MM']$.

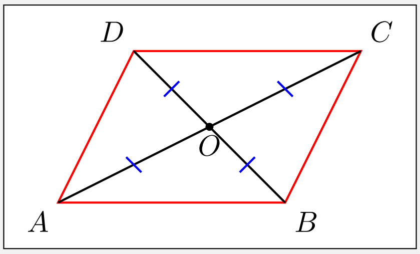


III - The parallelogram

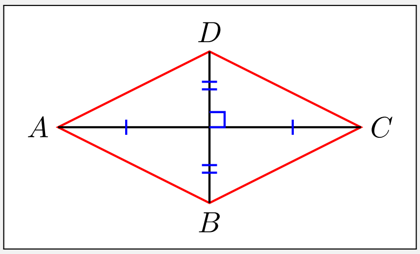
Definition 5 : A **parallelogram** is a quadrilateral whose opposite sides are parallel.



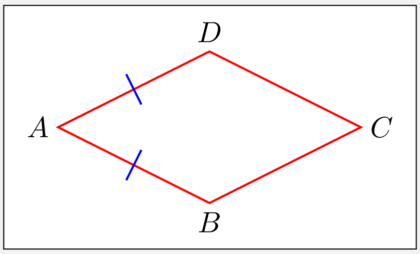
Property 3 : If the diagonals of a quadrilateral have the same midpoint then it is a parallelogram.



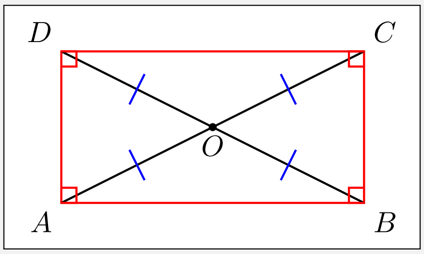
Property 4 : If a parallelogram has perpendicular diagonals then it is a rhombus.



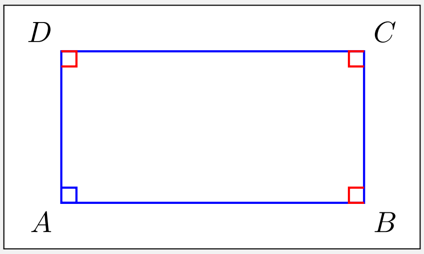
Property 5 : If a parallelogram has two consecutive sides of the same length then it is a rhombus.



Property 6 : If a parallelogram has diagonals of the same length then it is a rectangle.

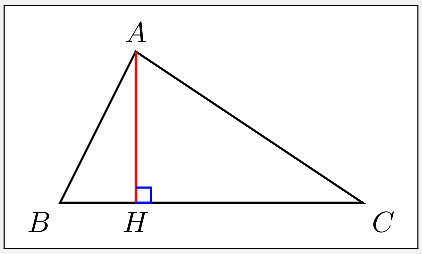


Property 7 : If a parallelogram has a right angle then it is a rectangle.

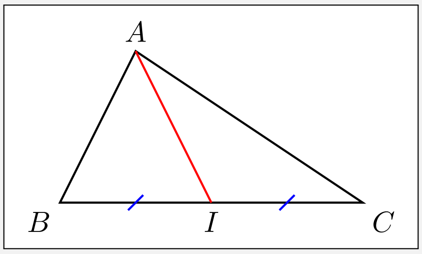


IV - The triangle

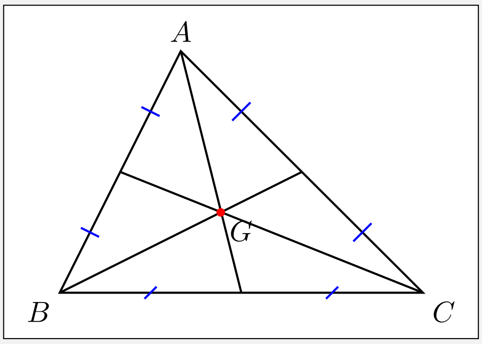
Definition 6 : The **altitude** of a triangle is a line that passes through a vertex of the triangle and is perpendicular to the side opposite this vertex.



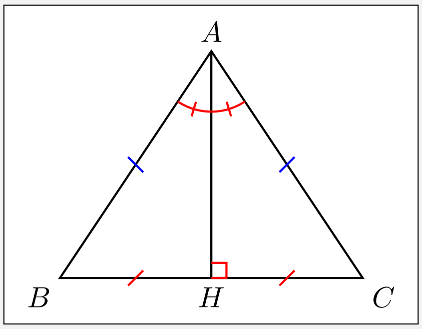
Definition 7 : In a triangle, a **median** is a line that passes through a vertex and the midpoint of the side opposite this vertex.



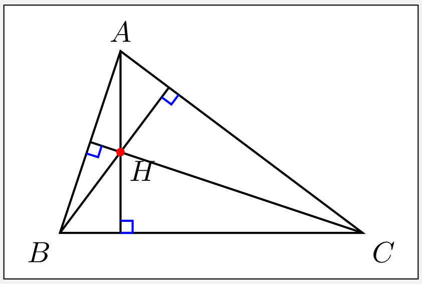
Property 8 : The three medians of a triangle are concurrent.
Their point of intersection is called the **centroid** of the triangle.



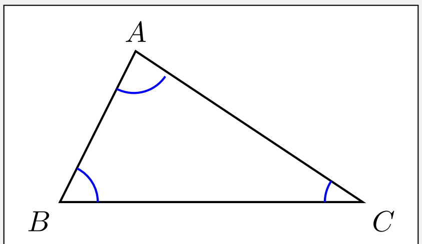
Property 9 : If a triangle is isosceles then the median, the altitude and the angle bisector passing through the principal vertex coincide with the perpendicular bisector of the base.



Property 10 : The three altitudes of a triangle are concurrent.
Their point of intersection is called the **orthocenter** of the triangle.



Property 11 : The sum of the measures of the angles of a triangle is equal to $180°$.

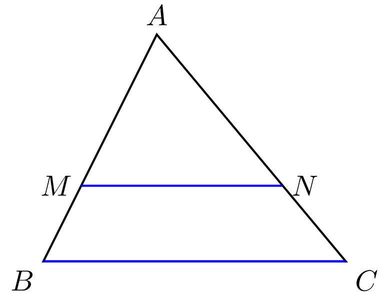


$$\hat{A} + \hat{B} + \hat{C} = 180°$$

Property 12 : Thales's Theorem

Let A, B, C, M and N be distinct points in the plane.

If $\begin{cases} (MB) \text{ and } (NC) \text{ intersect at } A \\ \text{and } (MN) \parallel (BC) \end{cases}$ then $\dfrac{AM}{AB} = \dfrac{AN}{AC} = \dfrac{MN}{BC}$

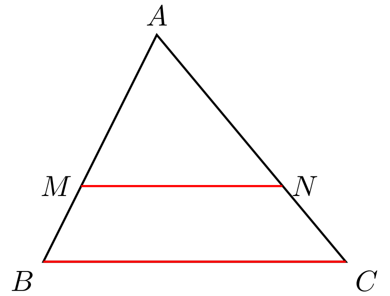


$$\frac{AM}{AB} = \frac{AN}{AC} = \frac{MN}{BC}$$

Property 13 : Converse of Thales's Theorem

Let A, B, C, M and N be distinct points in the plane.

If $\begin{cases} A, B, M \text{ and } A, C, N \text{ are aligned in the same order} \\ \dfrac{AM}{AB} = \dfrac{AN}{AC} \end{cases}$ then $(MN) \parallel (BC)$

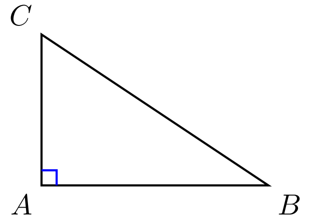


$$\frac{AM}{AB} = \frac{AN}{AC}$$

V - The right-angled triangle

Property 14 : Pythagorean Theorem

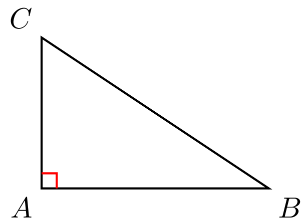
If a triangle is a right-angled triangle then the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the other two sides of the triangle.



$$BC^2 = AB^2 + AC^2$$

Property 15 : Converse of the Pythagorean Theorem

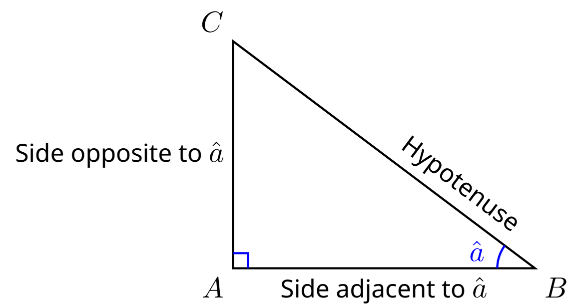
If, in a triangle, the square of the length of the longest side is equal to the sum of the squares of the lengths of the other two sides then this triangle is a right-angled triangle.



$$BC^2 = AB^2 + AC^2$$

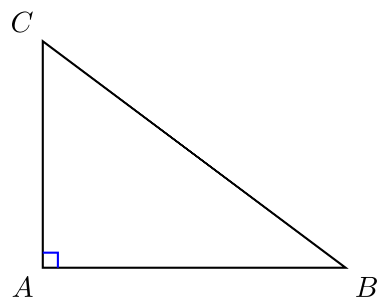
Definition 8 : Let $\hat{a}$ be an acute vertex angle of a right-angled triangle :

- We call the **cosine** of angle $\hat{a}$, denoted $\cos \hat{a}$, the ratio $\cos \hat{a} = \dfrac{\text{length of the side adjacent to } \hat{a}}{\text{length of the hypotenuse}}$
- We call the **sine** of angle $\hat{a}$, denoted $\sin \hat{a}$, the ratio $\sin \hat{a} = \dfrac{\text{length of the side opposite to } \hat{a}}{\text{length of the hypotenuse}}$
- We call the **tangent** of angle $\hat{a}$, denoted $\tan \hat{a}$, the ratio $\tan \hat{a} = \dfrac{\text{length of the side opposite to } \hat{a}}{\text{length of the side adjacent to } \hat{a}}$



Property 16 : In the triangle ABC right-angled at A :

$$(\cos \hat{B})^2 + (\sin \hat{B})^2 = 1$$



$$(\cos \hat{B})^2 + (\sin \hat{B})^2 = 1$$

Démonstration :

$$(\cos \hat{B})^2 + (\sin \hat{B})^2 = \left(\frac{AB}{BC}\right)^2 + \left(\frac{AC}{BC}\right)^2 = \frac{AB^2}{BC^2} + \frac{AC^2}{BC^2} = \frac{AB^2 + AC^2}{BC^2}$$

now, according to the Pythagorean theorem, $BC^2 = AB^2 + AC^2$

so $(\cos \hat{B})^2 + (\sin \hat{B})^2 = \dfrac{BC^2}{BC^2} = 1$

VI - Distances and circle

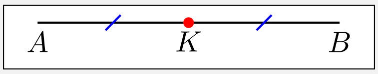
1 - Distances

Property 17 : Let A, B and M be three points in the plane.

- If $M \notin [AB]$ then $AM + MB > AB$
- If $M \in [AB]$ then $AM + MB = AB$

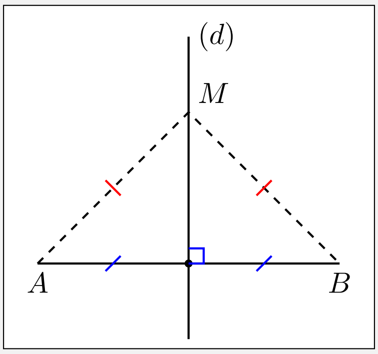
Definition 9 : The **midpoint** of a segment is the point on the segment that is equidistant from the ends of the segment.

$$K \text{ midpoint of } [AB] \iff \begin{cases} K \in [AB] \\ AK = KB \end{cases}$$



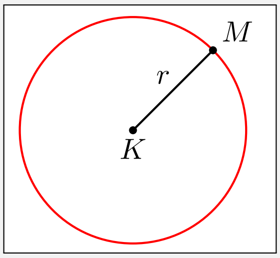
Property 18 :

- If a point belongs to the perpendicular bisector of a segment then it is equidistant from the ends of this segment.
- If a point is equidistant from the ends of a segment then it belongs to the perpendicular bisector of this segment.

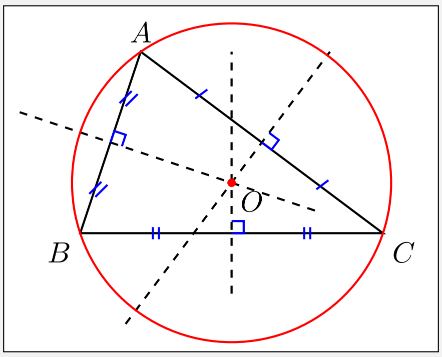


2 - The circle

Definition 10 : We call the **circle with center K and radius $r > 0$** the set of points M such that : $KM = r$.



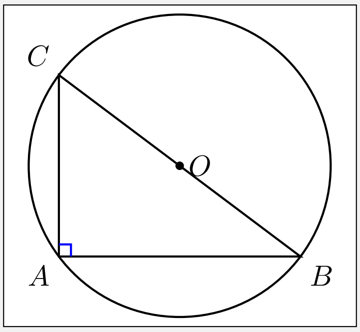
Property 19 : The 3 perpendicular bisectors of a triangle are concurrent (they intersect at the same point).
Their point of intersection is the center of the circle passing through the 3 vertices of the triangle, called the **circumscribed circle** of the triangle.



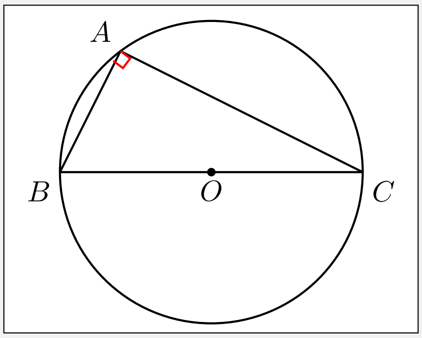
Proof :

Let ABC be any triangle.
Let I, J and K be the respective midpoints of the segments $[AB]$, $[BC]$ and $[AC]$.
Let O be the center of the circumscribed circle of the triangle ABC, hence $OA = OB = OC$.
$OA = OB$ so O belongs to the perpendicular bisector of $[AB]$.
$OB = OC$ so O belongs to the perpendicular bisector of $[BC]$.
$OA = OC$ so O belongs to the perpendicular bisector of $[AC]$.
Thus the 3 perpendicular bisectors of the triangle ABC are concurrent at O, the center of the circumscribed circle of the triangle ABC.

Property 20 : If a triangle is a right-angled triangle then its circumscribed circle has the hypotenuse of the triangle as its diameter.



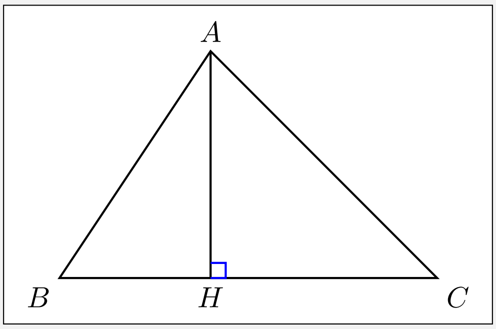
Property 21 : If one of the sides of a triangle is a diameter of its circumscribed circle then this triangle is a right-angled triangle and has this side as its hypotenuse.



3 - Area of the triangle

Property 22 : If the altitude (AH) of a triangle ABC intersects the line (BC) at H then :

$$\text{Area}(ABC) = \frac{AH \times BC}{2}$$



COORDINATES AND DISTANCE

I – Orthonormal coordinate system of the plane

Definition 1 : An **orthonormal coordinate system of the plane** is a triplet of points (O, I, J) forming an isosceles right-angled triangle with principal vertex O.

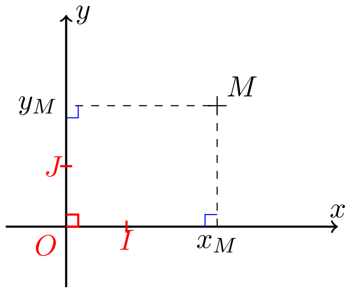
Vocabulary : The point O is called the **origin of the coordinate system**.
The line (OI) is called the **abscissa axis** and the distance OI defines the unit on this axis.
The line (OJ) is called the **ordinate axis** and the distance OJ defines the unit on this axis.

Note : This coordinate system is "ortho" (right) because the lines (OI) and (OJ) are perpendicular.
This coordinate system is "normed" because $OI = OJ$.

In the coordinate system (O, I, J) :

- x_M is called the **abscissa** of point M.
- y_M is called the **ordinate** of point M.
- $(x_M; y_M)$ is called the pair of **coordinates** of point M.

We denote : $\boldsymbol{M(x_M; y_M)}$



II – Distance between 2 points in an orthonormal coordinate system

Property 1 : Let $A(x_A; y_A)$ and $B(x_B; y_B)$ be two points of the plane equipped with an orthonormal coordinate system (O, I, J).

The distance between points A and B is :

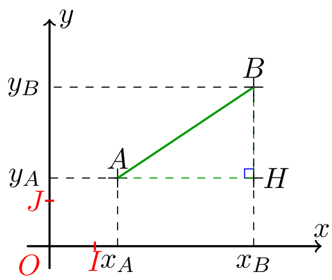
$$AB = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$$

Explanation of property 1

The coordinate system (O, I, J) is orthonormal so ABH is a right-angled triangle at H.
We apply the Pythagorean theorem :

$$\begin{aligned} AB^2 &= AH^2 + HB^2 \\ &= (x_B - x_A)^2 + (y_B - y_A)^2 \end{aligned}$$

Thus $AB = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$



Méthode : Calculating the distance between 2 points

In an orthonormal coordinate system (O, I, J), we are given : $C(-1; 2)$ and $D(3; -4)$.
Calculate the distance CD.

We apply property 1 :

$$\begin{aligned} CD &= \sqrt{(x_D - x_C)^2 + (y_D - y_C)^2} \\ &= \sqrt{(3 - (-1))^2 + (-4 - 2)^2} \\ &= \sqrt{52} \end{aligned}$$

If points C and D have been placed in an orthonormal coordinate system (O, I, J) such that $OI = OJ = 1$ cm, then we can verify the length CD on the graph by measuring ($CD \approx 7.2$ cm).

III – Coordinates of the midpoint of a segment in a coordinate system

Property 2 : Let $A(x_A; y_A)$ and $B(x_B; y_B)$ be two points of the plane equipped with a coordinate system (O, I, J).
The midpoint of the segment $[AB]$ has coordinates :

$$\left(\frac{x_A + x_B}{2}; \frac{y_A + y_B}{2}\right)$$

Explanation of property 2

(KB) and (LH) intersect at A, and $(KL) \parallel (BH)$.
We suppose that $x_A \neq x_B$ and $y_A \neq y_B$.
We apply Thales's theorem :

$$\frac{AK}{AB} = \frac{AL}{AH} = \frac{KL}{BH}$$

Now K is the midpoint of $[AB]$ so $\dfrac{AK}{AB} = \dfrac{1}{2}$

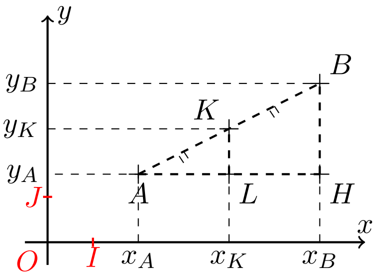
Thus $\dfrac{1}{2} = \dfrac{x_K - x_A}{x_B - x_A} = \dfrac{y_K - y_A}{y_B - y_A}$

$\dfrac{x_K - x_A}{x_B - x_A} = \dfrac{1}{2}$ and $\dfrac{y_K - y_A}{y_B - y_A} = \dfrac{1}{2}$

With a cross-multiplication, we obtain : $x_K - x_A = \dfrac{x_B - x_A}{2}$

hence $x_K = \dfrac{x_B - x_A}{2} + x_A = \dfrac{x_B - x_A}{2} + \dfrac{2x_A}{2} = \dfrac{x_B + x_A}{2}$

Similarly, we obtain : $y_K = \dfrac{y_B + y_A}{2}$



Note : Unlike property 1 which requires an orthonormal coordinate system, property 2 is true in any coordinate system.

Méthode : Calculating the coordinates of the midpoint of a segment

In a coordinate system (O, I, J), we are given : $C(-1; 2)$ and $D(3; -4)$.
Calculate the coordinates of the midpoint E of the segment $[CD]$.

We apply property 2 :

$$E\left(\frac{x_C + x_D}{2}; \frac{y_C + y_D}{2}\right)$$

$$E\left(\frac{-1 + 3}{2}; \frac{2 + (-4)}{2}\right)$$

$$E\,(1; -1)$$

GENERALITIES ON VECTORS

I – Representation of a vector

Property 1 :

$ABCD$ is a parallelogram is equivalent to $\begin{cases} ABCD \text{ is not crossed} \\ AB = CD \\ (AB) \parallel (CD) \end{cases}$

We group the information cited in property 1 into a new mathematical object : the vector.

Definition 1 : The **vector** $\overrightarrow{AB}$ is defined by :

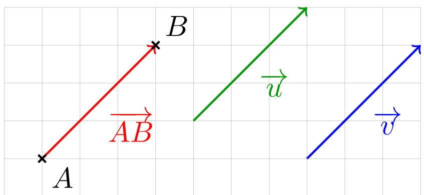
- its direction, the line (AB).
- its sense, from A to B.
- its length, the distance AB.

Representations of the vector $\overrightarrow{AB}$

A and B are 2 given points of the plane.
We represent the vector $\overrightarrow{AB}$ by an arrow going from A to B.

Any arrow parallel to the line (AB), of length AB, in the sense from A to B represents the vector $\overrightarrow{AB}$.

$$\overrightarrow{AB} = \overrightarrow{u} = \overrightarrow{v}$$

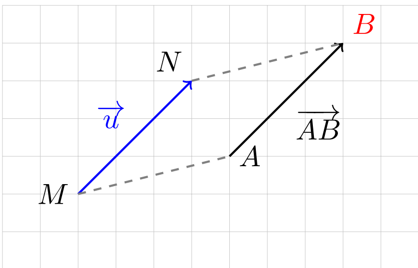


Property 2 : $\overrightarrow{AB} = \overrightarrow{CD}$ is equivalent to $ABDC$ is a parallelogram (possibly flat).

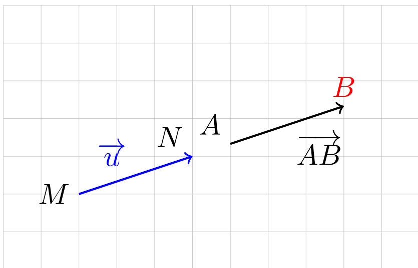
Méthode : Constructing a vector equal to a given vector

Let A be a given point and $\overrightarrow{u}$ a given vector. We call M the **origin** of the represented vector $\overrightarrow{u}$ and N its **extremity**.
To construct the point B such that $\overrightarrow{AB} = \overrightarrow{u}$, we draw the parallelogram $MNBA$.

A, M and N are not aligned



A, M and N are aligned



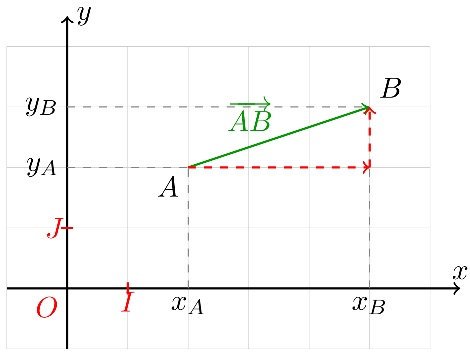
II – Coordinates of a vector

Coordinates of the vector $\overrightarrow{AB}$

The coordinates of a vector $\overrightarrow{AB}$ represent the path that links the origin A of the vector to its extremity B following the direction (Ox) then the direction (Oy).

In the coordinate system (O, I, J) opposite, we travel :

- $x_B - x_A$ following the direction (Ox),
- $y_B - y_A$ following the direction (Oy).



Property 3 : Let $A(x_A; y_A)$ and $B(x_B; y_B)$ be two given points in a coordinate system (O, I, J) of the plane.

The vector $\overrightarrow{AB}$ has coordinates $\begin{pmatrix} x_B - x_A \\ y_B - y_A \end{pmatrix}$.

The **zero vector** $\overrightarrow{AA}$ (which we also denote $\overrightarrow{0}$) has coordinates $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Property 4 : Two vectors are equal if and only if their coordinates are equal.

$$\overrightarrow{u}\begin{pmatrix} x \\ y \end{pmatrix} = \overrightarrow{v}\begin{pmatrix} x' \\ y' \end{pmatrix} \iff x = x' \text{ and } y = y'$$

I – Multiplication of a vector by a real number

Definition 2 : Let $\overrightarrow{AB}$ be a vector and k a real number (different from zero).

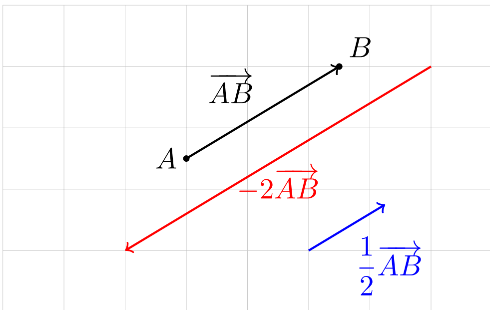
The **vector** $k\overrightarrow{AB}$ has :

- the direction (AB)
- the sense from A to B if $k > 0$ and the sense from B to A if $k < 0$
- the length kAB if $k > 0$ and $-kAB$ if $k < 0$

Méthode : Construction of the vector $k\overrightarrow{AB}$

Given the vector $\overrightarrow{AB}$, we draw the vectors $-2\overrightarrow{AB}$ and $\frac{1}{2}\overrightarrow{AB}$.

$$-2\overrightarrow{AB}\begin{cases}\text{direction } (AB)\\ \text{sense from B to A}\\ \text{length } 2AB\end{cases}$$



$$\frac{1}{2}\overrightarrow{AB}\begin{cases}\text{direction } (AB)\\ \text{sense from A to B}\\ \text{length } \frac{1}{2}AB\end{cases}$$

II – Collinear vectors

Definition 6 : We say that two **vectors** $\overrightarrow{u}$ and $\overrightarrow{v}$ are **collinear** when there exists a real number k such that $\overrightarrow{v} = k\overrightarrow{u}$.

Note : Two collinear vectors have the same direction.

Property 6 : Let A, B, C and D be distinct points of the plane.

$(AB) \parallel (CD)$ if and only if $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear.

Special case : $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are collinear if and only if the points A, B and C are aligned.

Property 7 : Let $\overrightarrow{u}\begin{pmatrix}x\\y\end{pmatrix}$ and $\overrightarrow{v}\begin{pmatrix}x'\\y'\end{pmatrix}$ be two given vectors in a coordinate system (O, I, J) of the plane.

The vectors $\overrightarrow{u}$ and $\overrightarrow{v}$ are collinear if and only if $xy' = x'y$.

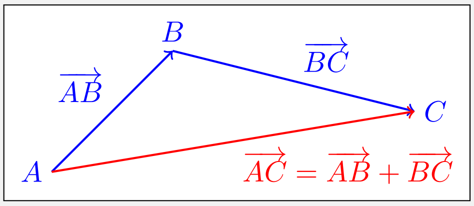
Démonstration :

Let $\overrightarrow{u}\begin{pmatrix}x\\y\end{pmatrix}$ and $\overrightarrow{v}\begin{pmatrix}x'\\y'\end{pmatrix}$ be two given vectors in a coordinate system (O, I, J) of the plane ($x \neq 0, y \neq 0$).

- If the vectors $\overrightarrow{u}$ and $\overrightarrow{v}$ are collinear then $\begin{cases}x' = kx\\ y' = ky\end{cases} \iff \begin{cases}k = \dfrac{x'}{x}\\ k = \dfrac{y'}{y}\end{cases}$ so $\dfrac{x'}{x} = \dfrac{y'}{y}$ so $x'y = xy'$.
- If $x'y = xy'$ then $\dfrac{x'}{x} = \dfrac{y'}{y}$ so there exists a real number k such that : $\begin{cases}k = \dfrac{x'}{x}\\ k = \dfrac{y'}{y}\end{cases} \iff \begin{cases}x' = kx\\ y' = ky\end{cases} \iff \overrightarrow{v} = k\overrightarrow{u}$.

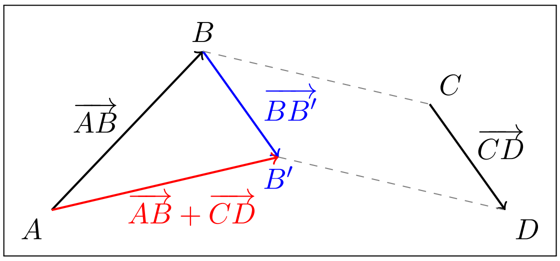
III – Addition of 2 vectors

Definition 7 : Let A, B, C be any three points, $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.



This equality is called **Chasles's relation**.

Méthode : Construction of the sum vector



$\overrightarrow{AB}$ and $\overrightarrow{CD}$ are two given vectors.
We construct B' such that $\overrightarrow{CD} = \overrightarrow{BB'}$.

We apply Chasles's relation :
$\overrightarrow{AB} + \overrightarrow{CD} = \overrightarrow{AB} + \overrightarrow{BB'} = \overrightarrow{AB'}$

Property 8 : Let $\overrightarrow{u}\begin{pmatrix} x \\ y \end{pmatrix}$ and $\overrightarrow{v}\begin{pmatrix} x' \\ y' \end{pmatrix}$ be two given vectors in a coordinate system (O, I, J) of the plane.

The vector $\overrightarrow{u} + \overrightarrow{v}$ has coordinates $\begin{pmatrix} x + x' \\ y + y' \end{pmatrix}$.

The vector $-\overrightarrow{u}$ has coordinates $\begin{pmatrix} -x \\ -y \end{pmatrix}$.

Definition 8 : The vector $\overrightarrow{BA}$ is called the **opposite of the vector** $\overrightarrow{AB}$. We denote : $\overrightarrow{BA} = -\overrightarrow{AB}$.

(it has the same direction, the same length but the opposite sense.)

Note : We find a relation already known for real numbers : $\overrightarrow{AB} + (-\overrightarrow{AB}) = \overrightarrow{AB} + \overrightarrow{BA} = \overrightarrow{AA} = \overrightarrow{0}$

If we add a vector and its opposite, we obtain the zero vector.

Definition 9 : Subtracting a vector is equivalent to adding its opposite : $\overrightarrow{u} - \overrightarrow{v} = \overrightarrow{u} + (-\overrightarrow{v})$.

Application :

Let A, B, C and D be four points of the plane. Simplify the following vector expression :

$$\overrightarrow{u} = \overrightarrow{AB} - \overrightarrow{CD} - \overrightarrow{AC} + \overrightarrow{BD}$$

We transform the subtractions into additions of the opposite, then we reorganize the terms to use Chasles's relation :

$$\begin{aligned}
\overrightarrow{u} &= \overrightarrow{AB} + \overrightarrow{DC} + \overrightarrow{CA} + \overrightarrow{BD} \\
\overrightarrow{u} &= (\overrightarrow{CA} + \overrightarrow{AB}) + (\overrightarrow{BD} + \overrightarrow{DC}) \\
\overrightarrow{u} &= \overrightarrow{CB} + \overrightarrow{BC} \\
\overrightarrow{u} &= \overrightarrow{CC} \\
\overrightarrow{u} &= \overrightarrow{0}
\end{aligned}$$

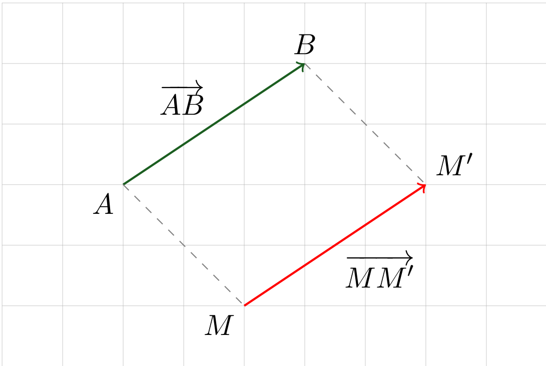
TRANSFORMATIONS OF THE PLANE

I – Translation

Definition 1 : Let A and B be two given points of the plane. The **translation by vector $\overrightarrow{AB}$** associates with any point M the point M' such that $\overrightarrow{MM'} = \overrightarrow{AB}$.

Méthode : Construction of the image of a point by a translation

Let A, B and M be three given points of the plane. We call M' the image of M by the translation by vector $\overrightarrow{AB}$. According to the definition $\overrightarrow{MM'} = \overrightarrow{AB}$ so it is sufficient to construct the parallelogram $ABM'M$.



If point M is on the line (AB) then the parallelogram $ABM'M$ is flat.

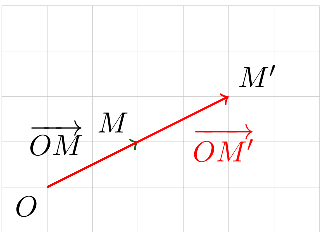
II – Homothety

Definition 2 : Let O be a given point of the plane and k a given real number. The **homothety of center O and ratio k** associates with any point M the point M' such that $\overrightarrow{OM'} = k\overrightarrow{OM}$.

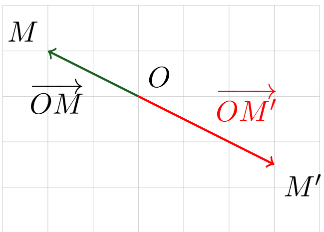
Méthode : Construction of the image of a point by a homothety

Let O be a point of the plane and k a real number given. We call M' the image of M by the homothety of center O and ratio k. According to the definition $\overrightarrow{OM'} = k\overrightarrow{OM}$.

$k = 2$



$k = -1.5$



III – Central symmetry

Definition 3 : Let O be a given point of the plane. The **central symmetry of center O** associates with any point M the point M' such that $\overrightarrow{OM'} = \overrightarrow{MO}$.

Notes :

- The equality $\overrightarrow{OM'} = \overrightarrow{MO}$ means that point O is the midpoint of the segment $[MM']$.
- Central symmetry is a homothety of ratio -1. Indeed, $\overrightarrow{OM'} = \overrightarrow{MO} \iff \overrightarrow{OM'} = -1 \times \overrightarrow{OM}$.

IV – Coordinates and transformations

Coordinates of the image of a point by a translation

Let $\overrightarrow{u}\begin{pmatrix}a\\b\end{pmatrix}$ be a vector and $M(x;y)$ a point given in a coordinate system (O,I,J).

Let $M'(x';y')$ be the image of M by the translation by vector $\overrightarrow{u}$ then $\overrightarrow{MM'}=\overrightarrow{u}$

$$\begin{pmatrix}x'-x\\y'-y\end{pmatrix}=\begin{pmatrix}a\\b\end{pmatrix}\iff\begin{cases}x'-x=a\\y'-y=b\end{cases}\iff\begin{cases}x'=x+a\\y'=y+b\end{cases}$$

Property 1 : Let $\overrightarrow{u}\begin{pmatrix}x\\y\end{pmatrix}$ be a vector and k a real number given in a coordinate system (O,I,J) of the plane.

The vector $k\overrightarrow{u}$ has coordinates $\begin{pmatrix}kx\\ky\end{pmatrix}$.

Coordinates of the image of a point by a homothety

Let $\Omega(a;b)$ be a point and $M(x;y)$ points given in a coordinate system (O,I,J), and k a given real number.
Let $M'(x';y')$ be the image of M by the homothety of center Ω and ratio k then $\overrightarrow{\Omega M'}=k\overrightarrow{\Omega M}$

$$\begin{pmatrix}x'-a\\y'-b\end{pmatrix}=k\begin{pmatrix}x-a\\y-b\end{pmatrix}\iff\begin{cases}x'-a=k(x-a)\\y'-b=k(y-b)\end{cases}\iff\begin{cases}x'=kx+a(1-k)\\y'=ky+b(1-k)\end{cases}$$

LINES OF THE PLANE

I – Direction vector of a line

Definition 1 : Let A and B be two distinct points of a line (d).
Any vector collinear with the vector $\overrightarrow{AB}$ is called a **direction vector** of the line (d).

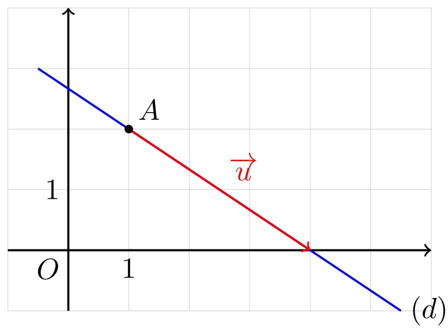
Méthode : Drawing a line given by a point and a direction vector

In a coordinate system (O, I, J), construct the line (d) passing through point $A(1; 2)$, with direction vector $\overrightarrow{u}\begin{pmatrix}3\\-2\end{pmatrix}$.

We place A.

We draw $\overrightarrow{u}$ with origin A.

We draw (d) passing through A and directed by $\overrightarrow{u}$.



II – Cartesian equation of a line

Property 1 : Any line (d) of the plane has an equation of the form $ax + by + c = 0$ with $a, b, c \in \mathbb{R}$ and $(a, b) \neq (0, 0)$.
This equation is called a **Cartesian equation** of the line (d).

Démonstration :

Let (d) be a line passing through a point $A(x_A; y_A)$ and with direction vector $\overrightarrow{u}\begin{pmatrix}\alpha\\\beta\end{pmatrix}$.

$M(x; y) \in (d) \iff \overrightarrow{AM}\begin{pmatrix}x - x_A\\y - y_A\end{pmatrix}$ and $\overrightarrow{u}\begin{pmatrix}\alpha\\\beta\end{pmatrix}$ are collinear
$\iff x_{\overrightarrow{AM}} \times y_{\overrightarrow{u}} - y_{\overrightarrow{AM}} \times x_{\overrightarrow{u}} = 0 \iff (x - x_A) \times \beta - (y - y_A) \times \alpha = 0$
$\iff \beta x - \beta x_A - \alpha y + \alpha y_A = 0 \iff \beta x - \alpha y - \beta x_A + \alpha y_A = 0$
$\iff ax + by + c = 0$ with $a = \beta$, $b = -\alpha$ and $c = -\beta x_A + \alpha y_A$.

Méthode : Determining the Cartesian equation of a line passing through 2 points

Determine a Cartesian equation of the line (AB) with $A(-1; 4)$ and $B(3; 2)$.

We calculate the coordinates of $\overrightarrow{AB}\begin{pmatrix}x_B - x_A\\y_B - y_A\end{pmatrix}$ hence $\overrightarrow{AB}\begin{pmatrix}4\\-2\end{pmatrix}$.

Let $M(x; y)$, we calculate the coordinates of $\overrightarrow{AM}\begin{pmatrix}x_M - x_A\\y_M - y_A\end{pmatrix}$ hence $\overrightarrow{AM}\begin{pmatrix}x + 1\\y - 4\end{pmatrix}$.

$M(x; y) \in (AB) \iff \overrightarrow{AM}$ and $\overrightarrow{AB}$ are collinear
$\iff x_{\overrightarrow{AM}} \times y_{\overrightarrow{AB}} - y_{\overrightarrow{AM}} \times x_{\overrightarrow{AB}} = 0 \iff (x + 1) \times (-2) - (y - 4) \times 4 = 0$
$\iff -2x - 2 - 4y + 16 = 0 \iff -2x - 4y + 14 = 0 \iff -x - 2y + 7 = 0.$

A Cartesian equation of the line (AB) is $-x - 2y + 7 = 0$.

Special case :

– The equation of the abscissa axis (Ox) is $y = 0$.
– The equation of the ordinate axis (Oy) is $x = 0$.

Property 2 : The vector $\overrightarrow{u}\begin{pmatrix}-b\\a\end{pmatrix}$ is a direction vector of the line $(d) : ax + by + c = 0$ with $(a, b) \neq (0, 0)$.

Démonstration :

To shorten the proof, we limit ourselves to the case where $a \neq 0$ and $b \neq 0$.
The points $A\left(0;\dfrac{-c}{b}\right)$ and $B\left(\dfrac{-c}{a};0\right)$ belong to the line (d) : $ax+by+c=0$.
Hence $\overrightarrow{AB}\begin{pmatrix}\dfrac{-c}{a}\\ \dfrac{-c}{b}\end{pmatrix}$ is a direction vector of (d). (pay attention to the sign of the ordinate of B)

$x_{\overrightarrow{u}} \times y_{\overrightarrow{AB}} - y_{\overrightarrow{u}} \times x_{\overrightarrow{AB}} = -b \times \left(\dfrac{-c}{b}\right) - a \times \left(\dfrac{-c}{a}\right) = c - c = 0$ so $\overrightarrow{u}\begin{pmatrix}-b\\ a\end{pmatrix}$ and $\overrightarrow{AB}$ are collinear.
Thus $\overrightarrow{u}$ is a direction vector of (d).

III – Reduced equation of a line

Property 3 :

- Any line not parallel to the ordinate axis has a reduced equation of the form $y = mx + p$ with $m, p \in \mathbb{R}$.
- Any line parallel to the ordinate axis has an equation of the form $x = k$ with $k \in \mathbb{R}$.

Démonstration :

In a coordinate system (O, I, J), any line (d) has a Cartesian equation of the form $ax + by + c = 0$ with $a, b, c \in \mathbb{R}$, $(a, b) \neq (0, 0)$. $\overrightarrow{u}\begin{pmatrix}-b\\ a\end{pmatrix}$ is a direction vector of (d).

The vector $\overrightarrow{j}\begin{pmatrix}0\\ 1\end{pmatrix}$ is a direction vector of the ordinate axis.

$(d) \parallel (Oy) \iff \overrightarrow{j}$ and $\overrightarrow{u}$ are collinear $\iff x_{\overrightarrow{j}} \times y_{\overrightarrow{u}} - y_{\overrightarrow{j}} \times x_{\overrightarrow{u}} = 0 \iff 0 \times a - 1 \times (-b) = 0 \iff b = 0.$

- If $b \neq 0$ then $\overrightarrow{j}$ and $\overrightarrow{u}$ are not collinear so (d) is not parallel to the ordinate axis and $ax + by + c = 0 \iff by = -ax - c \iff y = -\dfrac{a}{b}x - \dfrac{c}{b}$ so $y = mx + p$ with $m = -\dfrac{a}{b}$ and $p = -\dfrac{c}{b}$.
- If $b = 0$ then $\overrightarrow{j}$ and $\overrightarrow{u}$ are collinear so (d) is parallel to the ordinate axis and $ax + 0y + c = 0 \iff ax = -c \iff x = -\dfrac{c}{a}$ so $x = k$ with $k = -\dfrac{c}{a}$.

Vocabulary :

- The real number m is called the **slope** of the line.
- The real number p is called the **y-intercept**; it is the ordinate of the intersection point of the line with the ordinate axis.

Property 4 : Let $A(x_A; y_A)$ and $B(x_B; y_B)$ be two points of the plane such that $x_A \neq x_B$.

- The slope of the line (AB) is $m = \dfrac{y_B - y_A}{x_B - x_A}$.
- A direction vector of (AB) is $\overrightarrow{u}\begin{pmatrix}1\\ m\end{pmatrix}$.

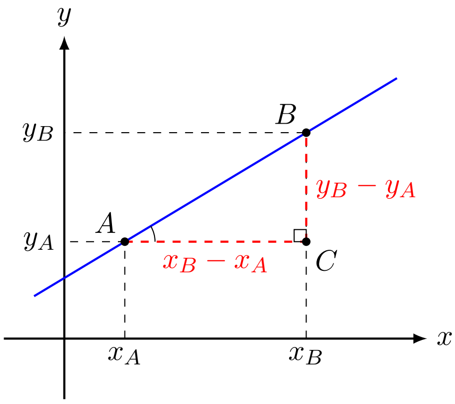
Démonstration :

- If $x_A \neq x_B$ then the line (AB) is not parallel to the ordinate axis so it has a reduced equation of the form $y = mx+p$ with $m, p \in \mathbb{R}$.
  $A \in (AB) \iff y_A = mx_A + p$ and $B \in (AB) \iff y_B = mx_B + p$
  hence $p = y_A - mx_A = y_B - mx_B \iff mx_B - mx_A = y_B - y_A \iff m(x_B - x_A) = y_B - y_A$
  now $x_A \neq x_B$ so $m = \dfrac{y_B - y_A}{x_B - x_A}$.
- $y = mx + p \iff mx - y + p = 0$ so $\overrightarrow{u}\begin{pmatrix}1\\ m\end{pmatrix}$ is a direction vector of (AB).

Graphical interpretation of the slope

(O, I, J) is an orthonormal coordinate system.
The triangle ABC is right-angled at C.
$\tan(\widehat{BAC}) = \dfrac{CB}{CA} = \dfrac{y_B - y_A}{x_B - x_A} = m$
so the slope m of the line determines the steepness of the line.

If $m > 0$ then the line "goes up".
If $m < 0$ then the line "goes down".
If $m = 0$ then the line is parallel to (Ox).



Property 5 : The lines $(d) : y = mx + p$ and $(d') : y = m'x + p'$ are parallel if and only if $m = m'$.

Démonstration :

If $(d) : y = mx + p$ and $(d') : y = m'x + p'$ then $\vec{u}\begin{pmatrix}1\\m\end{pmatrix}$ is a direction vector of (d) and $\vec{u'}\begin{pmatrix}1\\m'\end{pmatrix}$ is a direction vector of (d').
$(d) \parallel (d') \iff \vec{u}$ and $\vec{u'}$ are collinear $\iff x_{\vec{u}} \times y_{\vec{u'}} - y_{\vec{u}} \times x_{\vec{u'}} = 0 \iff 1 \times m' - m \times 1 = 0 \iff m = m'$.

PERCENTAGES

I - Applying a percentage

Property 1 : Applying $t\ \%$ to a number is multiplying this number by $\dfrac{t}{100}$.

Méthode : Applying a percentage

Calculate the amount of a $30\ \%$ discount on a price of 15 €.

Initial price : $P_0 = 15$ € Discount percentage : $t = 30$ (%) Discount amount : R

$$R = P_0 \times \frac{t}{100} = 15 \times \frac{30}{100} = 4.5 \text{ €}$$

II - Calculating a percentage

Property 2 : As a percentage, a part q of a quantity Q is equal to $\dfrac{q}{Q} \times 100$.

Méthode : Calculating a percentage

Calculate the percentage that a discount of 5 € represents on a total amount of 40 €.

Part in €	$q = 5$	x
Total in €	$Q = 40$	100

$$x = \frac{q}{Q} \times 100 = \frac{5}{40} \times 100 = 12.5 \text{ (\%)}$$

III - Percentage evolution

Property 3 :

- Increasing a quantity by $t\ \%$ is equivalent to multiplying it by $\left(1 + \dfrac{t}{100}\right)$.
- Reducing a quantity by $t\ \%$ is equivalent to multiplying it by $\left(1 - \dfrac{t}{100}\right)$.

Démonstration : Percentage increase

We call Q_0 the initial quantity and Q_1 the final quantity increased by $t\ \%$.

According to property 1, the increase is equal to $\dfrac{t}{100} \times Q_0$ so $Q_1 = Q_0 + \dfrac{t}{100} \times Q_0$.

Factoring Q_0 in the previous expression, we get :

$$Q_1 = 1 \times Q_0 + \frac{t}{100} \times Q_0 = \left(1 + \frac{t}{100}\right) \times Q_0$$

Méthode : Percentage reduction

Calculate the price of an article costing 50 € on sale for $20\ \%$.

Initial price : $P_0 = 50$ € Discount percentage : $t = 20$ (%) Final price : P_1

$$P_1 = \left(1 - \frac{t}{100}\right) \times P_0 = \left(1 - \frac{20}{100}\right) \times 50 = 40 \text{ €}$$

Vocabulaire : The number t is called the **percentage of evolution** from Q_0 to Q_1.

EVOLUTION RATE

I - Evolution rate and evolution percentage

Definition 1 : The number T is called the **evolution rate** from the quantity Q_0 to the quantity Q_1 :

$$T = \frac{Q_1 - Q_0}{Q_0}$$

Property 1 : If t is the evolution percentage from Q_0 to Q_1 and T the evolution rate from Q_0 to Q_1 then :

$$T = \frac{t}{100}$$

Démonstration : Evolution rate and evolution percentage

We call $Q_0 \neq 0$ the initial quantity and Q_1 the final quantity increased by $t\,\%$.

According to property 1, the increase is equal to $\frac{t}{100} \times Q_0$ so $Q_1 = Q_0 + \frac{t}{100} \times Q_0$.

$$\begin{aligned} Q_1 = Q_0 + \frac{t}{100} \times Q_0 &\iff Q_1 - Q_0 = \frac{t}{100} \times Q_0 \\ &\iff (Q_1 - Q_0) \times \frac{1}{Q_0} = \frac{t}{100} \times \cancel{Q_0} \times \frac{1}{\cancel{Q_0}} \\ &\iff \frac{Q_1 - Q_0}{Q_0} = \frac{t}{100} \quad \text{so} \quad T = \frac{t}{100} \end{aligned}$$

Consequence : Using property 3, we obtain : $Q_1 = (1 + T)Q_0$

Méthode : Evolution rate and evolution percentage

Calculate the evolution rate and the evolution rate as a percentage of a price that goes from 20 € to 45 €.

$$P_0 = 20\text{ €} \xrightarrow{\quad T \quad} P_1 = 45\text{ €}$$

$$T = \frac{P_1 - P_0}{P_0} = \frac{45 - 20}{20} = 1.25 \quad \text{and} \quad T = \frac{t}{100} \quad \text{so} \quad t = 100 \times T = 100 \times 1.25 = 125\ (\%)$$

Remarque :

- During an increase $T > 0$ and during a reduction $T < 0$.
- The quantities must be expressed in the same unit.

II - Successive evolutions

Property 2 :

- If T_1 is the evolution rate from Q_0 to Q_1, and T_2 the evolution rate from Q_1 to Q_2 then :

$$Q_2 = (1 + T_1)(1 + T_2)Q_0$$

- If T is the evolution rate from Q_0 to Q_2 then $1 + T = (1 + T_1)(1 + T_2)$.

Méthode : Overall evolution rate

An article increases by $20\,\%$ then decreases by $20\,\%$. Calculate its overall evolution percentage t.

$$\begin{tikzcd} P_0 \arrow[r, "T_1 = 0.2"] \arrow[rr, bend right, "T"'] & P_1 \arrow[r, "T_2 = -0.2"] & P_2 \end{tikzcd}$$

$$1 + T = (1 + T_1)(1 + T_2) = (1 + 0.2)(1 - 0.2) = 0.96 \quad \text{so} \quad T = 0.96 - 1 = -0.04$$

so $t = T \times 100 = -0.04 \times 100 = -4$ (%)

III - Reciprocal evolution

Definition 2 : The number T' is called the **reciprocal evolution rate** from the quantity Q_0 to the quantity Q_1 :

$$T' = \frac{Q_0 - Q_1}{Q_1}$$

Property 3 : If T is the evolution rate from Q_0 to Q_1 and T' the reciprocal evolution rate from Q_0 to Q_1 then :

$$1 + T' = \frac{1}{1+T}$$

Démonstration : Reciprocal evolution rate

$$1 + T' = \frac{1}{1+T} \iff (1+T') \times (1+T) = 1$$

$$\begin{aligned}
(1+T') \times (1+T) &= \left(1 + \frac{Q_0 - Q_1}{Q_1}\right)\left(1 + \frac{Q_1 - Q_0}{Q_0}\right) \\
&= \left(\frac{Q_1}{Q_1} + \frac{Q_0 - Q_1}{Q_1}\right)\left(\frac{Q_0}{Q_0} + \frac{Q_1 - Q_0}{Q_0}\right) \\
&= \left(\frac{\cancel{Q_1} + Q_0 - \cancel{Q_1}}{Q_1}\right)\left(\frac{\cancel{Q_0} + Q_1 - \cancel{Q_0}}{Q_0}\right) \\
&= \left(\frac{Q_0}{Q_1}\right)\left(\frac{Q_1}{Q_0}\right) = \frac{\cancel{Q_0} \times \cancel{Q_1}}{\cancel{Q_1} \times \cancel{Q_0}} = 1
\end{aligned}$$

Méthode : Reciprocal evolution rate

An article increases by $20\,\%$. Calculate the reciprocal evolution percentage t'.

$$P_0 \underset{\xleftarrow[T']{}}{\xrightarrow{T=0.2}} P_1$$

$$1 + T' = \frac{1}{1+T} = \frac{1}{1+0.2} = \frac{1}{1.2} \quad \text{so} \quad T' = \frac{1}{1.2} - 1 = -\frac{1}{6} \approx -0.17$$

so $t' = 100 \times T' \approx -17$ (%)

ONE-VARIABLE STATISTICAL SERIES

I - Measures of dispersion

Definition 1 : The **median** Me is the value that divides the series into 2 groups such that :

- at least $50\ \%$ of the values are less than or equal to Me
- at least $50\ \%$ of the values are greater than or equal to Me.

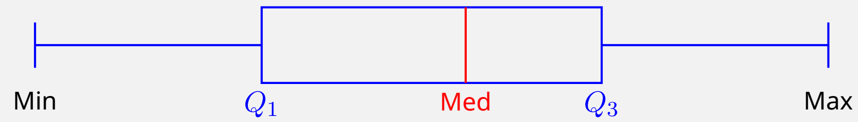
Remarque : The median is the central value of the series.

Definition 2 :

- The **first quartile** is the value Q_1 that divides the series into 2 groups such that :
  - at least $25\ \%$ of the values are less than or equal to Q_1
  - at least $75\ \%$ of the values are greater than or equal to Q_1.
- The **third quartile** is the value Q_3 that divides the series into 2 groups such that :
  - at least $75\ \%$ of the values are less than or equal to Q_3
  - at least $25\ \%$ of the values are greater than or equal to Q_3.

Definition 3 :

- The interval $[Q_1\ ;Q_3]$ is called the **interquartile interval**.
- The difference $Q_3 - Q_1$ is called the **interquartile range**.
- The diagram below, drawn to scale, is called a **box plot** :



Remarque : The interquartile range measures the dispersion of the series. Indeed, it contains about $50\ \%$ of the values of the series.

II - Measures of central tendency

Study of the statistical series below :

Value x_i	x_1	x_2	$\dots$	x_k
Frequency n_i	n_1	n_2	$\dots$	n_k

Definition 4 : The **mean** of a statistical series, denoted $\overline{x}$, is the number :

$$\overline{x} = \frac{n_1 \times x_1 + n_2 \times x_2 + \cdots + n_k \times x_k}{n_1 + n_2 + \cdots + n_k}$$

Definition 5 : The **standard deviation** of a statistical series, denoted σ, is the number :

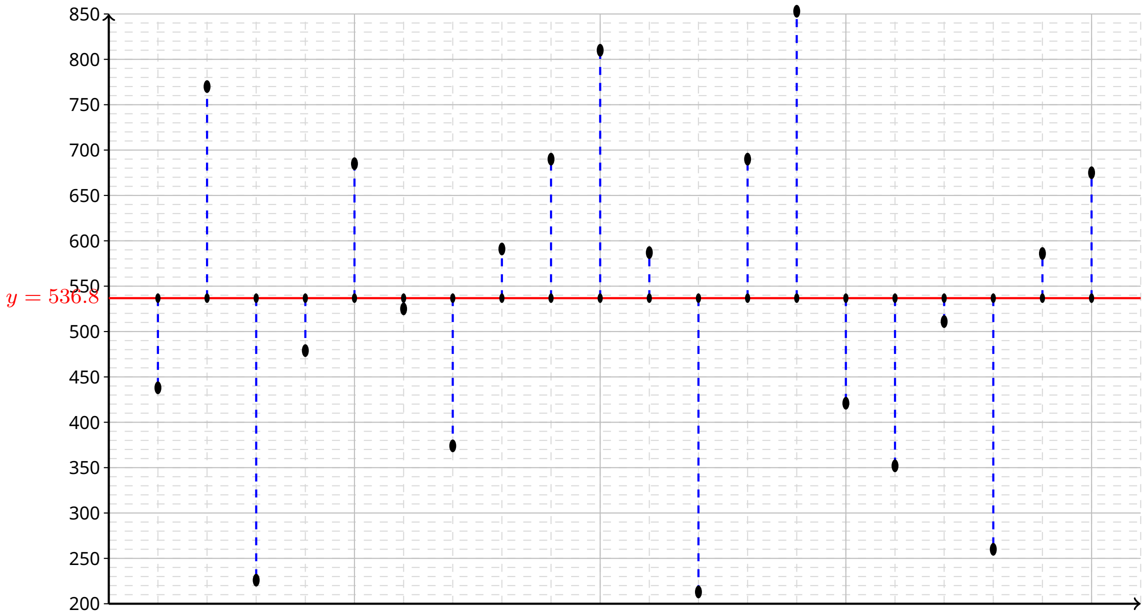
$$\sigma = \sqrt{\frac{n_1(x_1 - \overline{x})^2 + n_2(x_2 - \overline{x})^2 + \cdots + n_k(x_k - \overline{x})^2}{n_1 + n_2 + \cdots + n_k}}$$

Remarque : The standard deviation measures the dispersion of values around the mean.
The larger it is, the more the values are dispersed and the less the mean is representative of the values in the series.

Explication : of the standard deviation The accountant of a towing company recorded the mileage of each vehicle during one week.
We obtained the following numbers, in kilometers :

438	770	226	479	685	525	374	591	690	810
587	213	690	853	421	352	511	260	586	675

We have represented all the values of the series on the graph below, as well as the deviation between the mean and each value (blue dotted lines). The standard deviation calculates the average of the lengths of the blue dotted segments, i.e. the average deviation of the values from the mean.



Property 1 :

- If we add the same number a to all values of a statistical series with mean $\overline{x}$, then the mean of the resulting series is $\overline{x} + a$.
- If we multiply all values of a statistical series with mean $\overline{x}$ by the same number b, then the mean of the resulting series is $\overline{x} \times b$.

Démonstration :

- **We add a to all values of the series.**

Value x_i	$x_1 + a$	$x_2 + a$	$\dots$	$x_k + a$
Frequency n_i	n_1	n_2	$\dots$	n_k

Calculation of the mean :

$$\begin{aligned}
\overline{X} &= \frac{n_1 \times (x_1 + a) + n_2 \times (x_2 + a) + \cdots + n_k \times (x_k + a)}{n_1 + n_2 + \cdots + n_k} \\
&= \frac{n_1 x_1 + n_1 a + n_2 x_2 + n_2 a + \cdots + n_k x_k + n_k a}{n_1 + n_2 + \cdots + n_k} \\
&= \frac{n_1 x_1 + n_2 x_2 + \cdots + n_k x_k + n_1 a + n_2 a + \cdots + n_k a}{n_1 + n_2 + \cdots + n_k} \\
&= \frac{n_1 x_1 + n_2 x_2 + \cdots + n_k x_k}{n_1 + n_2 + \cdots + n_k} + \frac{n_1 a + n_2 a + \cdots + n_k a}{n_1 + n_2 + \cdots + n_k} \\
&= \overline{x} + \frac{(n_1 + n_2 + \cdots + n_k) \times a}{n_1 + n_2 + \cdots + n_k} \\
&= \overline{x} + \frac{\cancel{(n_1 + n_2 + \cdots + n_k)} \times a}{\cancel{(n_1 + n_2 + \cdots + n_k)} \times 1} \\
&= \overline{x} + a
\end{aligned}$$

- **We multiply all values of the series by b.**

Value x_i	$x_1 \times b$	$x_2 \times b$	$\dots$	$x_k \times b$
Frequency n_i	n_1	n_2	$\dots$	n_k

Calculation of the mean :

$$\begin{aligned}
\overline{Y} &= \frac{n_1 \times (x_1 \times b) + n_2 \times (x_2 \times b) + \cdots + n_k \times (x_k \times b)}{n_1 + n_2 + \cdots + n_k} \\
&= \frac{b \times (x_1 \times n_1) + b \times (x_2 \times n_2) + \cdots + b \times (x_k \times n_k)}{n_1 + n_2 + \cdots + n_k} \\
&= \frac{b \times (n_1 x_1 + n_2 x_2 + \cdots + n_k x_k)}{n_1 + n_2 + \cdots + n_k} \\
&= b \times \frac{n_1 x_1 + n_2 x_2 + \cdots + n_k x_k}{n_1 + n_2 + \cdots + n_k} \\
&= b \times \overline{x}
\end{aligned}$$

III - Cross-tabulation of two characters

A cross-tabulation of two characters contains the values of two characters and their frequencies.

Explication : Cross-tabulation of frequencies

In a class of 35 students, 16 ski, 11 snowboard, and 4 practice both sports.
We denote the characters A : "The student skis" and B : "The student snowboards".

	A	$\overline{A}$	Total
B	$A \cap B$: 4	$\overline{A} \cap B$: 7	B : 11
$\overline{B}$	$A \cap \overline{B}$: 12	$\overline{A} \cap \overline{B}$: 12	$\overline{B}$: 24
Total	A : 16	$\overline{A}$: 19	35

Definition 6 : The **marginal frequency of A**, denoted f_A, is :

$$f_A = \frac{\text{frequency of A}}{\text{total frequency}}$$

Explication : Calculation of marginal frequency

The marginal frequency of students who ski is $f_A = \dfrac{16}{35}$.

Definition 7 : The **conditional frequency of A given B**, denoted $f_B(A)$, is :

$$f_B(A) = \frac{\text{frequency of } A \cap B}{\text{frequency of B}}$$

Explication : Calculation of conditional frequency

The conditional frequency $f_B(A)$ is the frequency of students who ski among those who snowboard.

$$f_B(A) = \frac{4}{11}$$

EVENT AND SET LANGUAGE

I - Vocabulary

Definition 1 : An experiment is called a **random experiment** if it satisfies the following 2 conditions :

1 - All possible outcomes are known
2 - We do not know in advance which outcome will occur.

Study of a specific case :

We roll a fair 6-sided die numbered from 1 to 6. We note the number obtained on the top face.

- All possible outcomes are known : they are 1, 2, 3, 4, 5, 6.
- We cannot know in advance which outcome we will get.

It is therefore a random experiment.

Definition 2 : An **outcome** is a possible result of a random experiment.

In the example :

There are 6 outcomes : they are 1, 2, 3, 4, 5 and 6.

Definition 3 : The **universe** (or sample space), denoted Ω, is the set of all outcomes of a random experiment.

In the example :

The universe is $\Omega = \{1, 2, 3, 4, 5, 6\}$.

Definition 4 : An **event** is a set of outcomes.

In the example :

- Let A be the event "Obtaining an even number"; it is composed of the outcomes 2, 4 and 6.

$$A = \{2, 4, 6\}$$

- Let B be the event "Obtaining a number greater than or equal to 2"; it is composed of the outcomes 2, 3, 4, 5, 6.

$$B = \{2, 3, 4, 5, 6\}$$

II - Use of set language

Notation :

- An outcome e, also called an **elementary event**, is an **element** of Ω. We denote $e \in \Omega$.
- An event A is a **subset** of the universe Ω. We say that A is **included** in Ω and we denote $A \subset \Omega$.

Definition 5 : **Mutually exclusive events** are events that cannot occur simultaneously.

In the example :

Let C be the event "Rolling a 6"; $C = \{6\}$.
Let D be the event "Rolling a number strictly less than 3"; $D = \{1, 2\}$.
Events C and D have no common outcome; they are therefore mutually exclusive.

Notation :

- When 2 sets have no element in common, we say that their intersection is an empty set.
  The **empty set** is denoted $\varnothing$. The symbol for **intersection** is $\cap$.

- If 2 events C and D are mutually exclusive, we therefore denote $C \cap D = \varnothing$.

Vocabulaire :

- All outcomes belong to the universe Ω so we call Ω the **certain event**.

- The empty set $\varnothing$ contains no outcome so we call $\varnothing$ the **impossible event**.

Definition 6 : The **complementary event** of an event A, denoted $\overline{A}$, is the event composed of all outcomes not belonging to A.

In the example :

- Let A be the event "Rolling an even number" : $A = \{2, 4, 6\}$
  hence $\overline{A}$ is the event "Not rolling an even number" so $\overline{A} = \{1, 3, 5\}$.

- Let B be the event "Rolling a number greater than or equal to 2" : $B = \{2, 3, 4, 5, 6\}$
  hence $\overline{B}$ is the event "Rolling a one" so $\overline{B} = \{1\}$.

Property 1 : For all events A, B :

Event **A or** event **B** is the event $\mathbf{A} \cup \mathbf{B}$.

Event **A and** event **B** is the event $\mathbf{A} \cap \mathbf{B}$.

$$A \cap \overline{A} = \varnothing \quad \text{and} \quad A \cup \overline{A} = \Omega.$$

In the example :

Let A be the event "Rolling an even number"; $A = \{2, 4, 6\}$
and D the event "Rolling a number strictly less than 3"; $D = \{1, 2\}$.

A or D consists of the outcomes contained in A and in D; $A \cup D = \{1, 2, 4, 6\}$.

A and D consists of the outcomes common to A and D; $A \cap D = \{2\}$.

PROBABILITY CALCULATION

I - Probability of an event

1 - Concept of probability

The probability of an event is the frequency of occurrence of this event during a large number of repetitions of the same random experiment.

In the example :

If we roll a 6-sided die a very large number of times, we will observe that the frequency of occurrence of each face is equal to 1 out of 6.

Definition 1 :

During a random experiment, we say there is **equiprobability** when all outcomes have the same probability of occurring.

Property 1 :

In a situation of equiprobability, the probability of an event A is $p(A) = \dfrac{\text{number of outcomes in A}}{\text{number of outcomes in } \Omega}$.

In the example :

The universe is $\Omega = \{1, 2, 3, 4, 5, 6\}$ and contains 6 outcomes.

Let A be the event "Rolling an even number"; $A = \{2, 4, 6\}$ contains 3 outcomes.

$$p(A) = \frac{3}{6} = 0.5$$

Let B be the event "Rolling a number greater than or equal to 2"; $B = \{2, 3, 4, 5, 6\}$ contains 5 outcomes.

$$p(B) = \frac{5}{6} \approx 0.83$$

Consequence :

- The probability of an event is between 0 and 1.
- The sum of the probabilities of all outcomes is equal to 1.

2 - Probability distribution

Definition 2 :

Let Ω be the finite universe associated with a random experiment. Defining a probability distribution on Ω means giving the probabilities p_i of each outcome x_i of Ω.
We group these results in a table :

outcome x_i	x_1	x_2	$\dots$	x_n
probability p_i	p_1	p_2	$\dots$	p_n

In the example :

The probability of each outcome is equal to $\frac{1}{6}$.

outcome	1	2	3	4	5	6
probability	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

We indeed verify that :

- each probability has a value between 0 and 1.
- the sum of the probabilities of each outcome is equal to 1 $\left(\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}=\frac{6}{6}=1\right)$.

3 - Properties

Property 2 :

- For all events A and B, $p(A \cup B)=p(A)+p(B)-p(A \cap B)$.
- For any event A, $p(A)+p(\overline{A})=1$.

Remarque : If events A and B are mutually exclusive then $p(A \cup B)=p(A)+p(B)$.

In the example :

Let A be the event "Rolling an even number" : $A=\{2,4,6\}$ so $p(A)=\frac{3}{6}$

and D the event "Rolling a number strictly less than 3" : $D=\{1,2\}$ so $p(D)=\frac{2}{6}$

$A \cup D=\{1,2,4,6\}$ so $p(A \cup D)=\frac{4}{6}$

$A \cap D=\{2\}$ so $p(A \cap D)=\frac{1}{6}$

so $p(A)+p(D)-p(A \cap D)=\frac{3}{6}+\frac{2}{6}-\frac{1}{6}=\frac{4}{6}=p(A \cup D)$

II - Modeling with a table or a tree

1 - With a table

Two-way table :

We roll 2 identical dice and note the sum obtained. Describe the universe of this random experiment.
We note all the possible rolls when throwing the 2 dice in the table below.

die 1 \ die 2	**1**	**2**	**3**	**4**	**5**	**6**
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

There are 36 possible rolls. $\Omega = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

Probability of each outcome :
$p(2) = \frac{1}{36} = p(12)\,;\, p(3) = \frac{2}{36} = p(11)\,;\, p(4) = \frac{3}{36} = p(10)\,;\, p(5) = \frac{4}{36} = p(9)\,;\, p(6) = \frac{5}{36} = p(8)\,;\, p(7) = \frac{6}{36}.$

Probability distribution :

outcome	2	3	4	5	6	7	8	9	10	11	12
probability	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

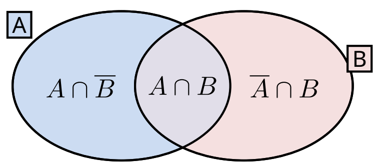
We notice that this is not a situation of equiprobability.

(Additive) two-way table :

In a class of 35 students, knowing that 16 ski, 11 snowboard and 4 practice both sports, how many students practice no sport?

We consider the events A : "Skiing" and B : "Snowboarding".

	A	$\overline{A}$	Total
B	$p(A \cap B) = \frac{4}{35}$	$p(\overline{A} \cap B) = \frac{7}{35}$	$p(B) = \frac{11}{35}$
$\overline{B}$	$p(A \cap \overline{B}) = \frac{12}{35}$	$p(\overline{A} \cap \overline{B}) = \frac{12}{35}$	$p(\overline{B}) = \frac{24}{35}$
Total	$p(A) = \frac{16}{35}$	$p(\overline{A}) = \frac{19}{35}$	1



$p(\overline{A} \cap \overline{B}) = \frac{12}{35}$ so there are 12 students who practice no sport.

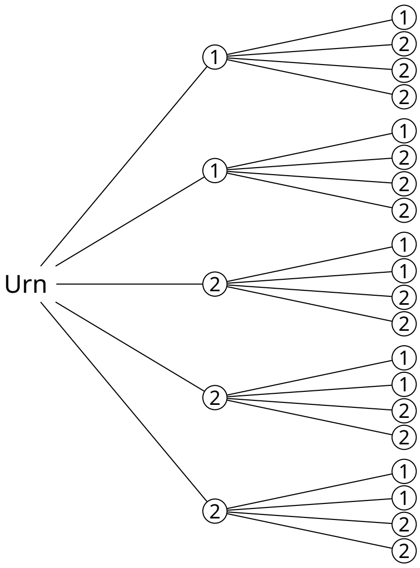
2 - With a simple tree

Simple tree (of choices) :

An urn contains 5 identical balls numbered 1 or 2. We successively draw two balls from the urn without replacement.

1) What is the probability p_1 of drawing 2 balls with the number 1 ?
2) What is the probability p_2 of drawing exactly one ball with the number 2 ?
3) What is the probability p_3 of drawing at least one ball with the number 2 ?

We can represent the situation with a simple (choice) tree :



In this tree, a path is composed of 2 branches and represents an outcome. There are a total of 20 equiprobable outcomes.

1) There are 2 paths for which the 2 balls have the number 1 :

$$p_1 = \frac{2}{20} = \frac{1}{10}$$

2) There are 12 paths for which exactly one ball has the number 2 :

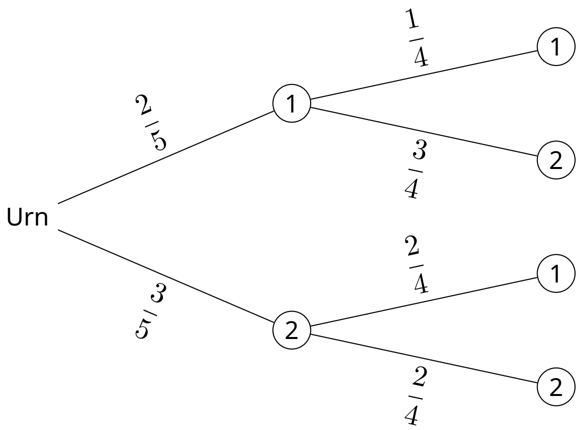
$$p_2 = \frac{12}{20} = \frac{3}{5}$$

3) There are 18 paths for which at least one ball has the number 2 :

$$p_3 = \frac{18}{20} = \frac{9}{10}$$

3 - With a weighted tree

In the previous example :



Property 3 :

- In a weighted tree, the probability of an event corresponding to a path is equal to the product of the probabilities of the branches that make up this path.
- The sum of the probabilities of the branches issued from the same node is equal to 1.

In the previous example :

$$p_1 = \frac{2}{5} \times \frac{1}{4} = \frac{2}{20} = \frac{1}{10} \qquad p_2 = \frac{2}{5} \times \frac{3}{4} + \frac{3}{5} \times \frac{2}{4} = \frac{6}{20} + \frac{6}{20} = \frac{12}{20} = \frac{3}{5}$$

$$p_3 = \frac{2}{5} \times \frac{3}{4} + \frac{3}{5} \times \frac{2}{4} + \frac{3}{5} \times \frac{2}{4} = \frac{6}{20} + \frac{6}{20} + \frac{6}{20} = \frac{18}{20} = \frac{9}{10}$$

III - Conditional probability

Definition 3 : Let A and B be two events with $p(A) \neq 0$. The **probability of B given that A has occurred** is the number :

$$p_A(B) = \frac{p(A \cap B)}{p(A)}$$

Notes : • The previous formula is also written $p(A \cap B) = p_A(B) \times p(A)$.
• The event $A \cap B$ occurs when event A occurs and event B occurs.

Property 4 : If $p(A) \neq 0$ and $p(B) \neq 0$ then $p(A \cap B) = p_A(B) \times p(A) = p_B(A) \times p(B)$.

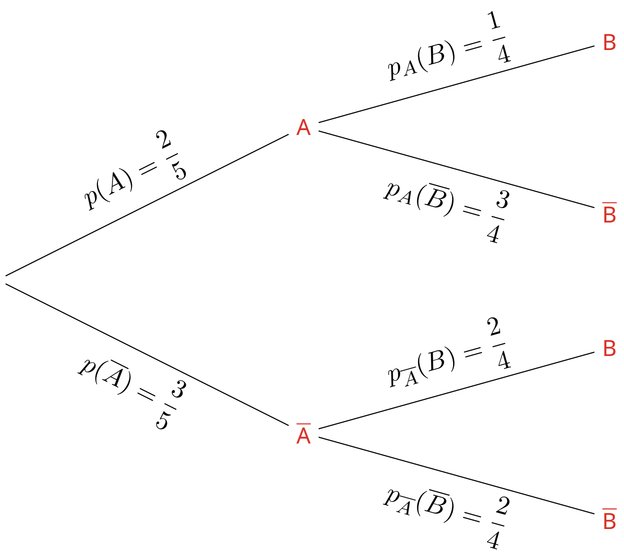
Application to the previous example :

We denote the events : • A : "The first drawn ball has the number 1."
• B : "The second drawn ball has the number 1."

1) What do the events $\overline{A}$ and $\overline{B}$ represent?
2) Redo the weighted tree with the notations introduced in this paragraph.
3) Write the literal formulas of p_1, p_2, p_3 calculated previously.

1) $\overline{A}$: "The first ball does not have the number 1." (so it has the number 2)
$\overline{B}$: "The second ball does not have the number 1." (so it has the number 2)

2) Weighted tree :



3) $p_1 = p(A \cap B) = p(A) \times p_A(B) = \frac{2}{5} \times \frac{1}{4} = \frac{2}{20} = \frac{1}{10}$

$$p_2 = p(A \cap \overline{B}) + p(\overline{A} \cap B) = p(A) \times p_A(\overline{B}) + p(\overline{A}) \times p_{\overline{A}}(B) = \frac{2}{5} \times \frac{3}{4} + \frac{3}{5} \times \frac{2}{4} = \frac{6}{20} + \frac{6}{20} = \frac{12}{20} = \frac{3}{5}$$

$$p_3 = p(A \cap \overline{B}) + p(\overline{A} \cap B) + p(\overline{A} \cap \overline{B}) = p(A) \times p_A(\overline{B}) + p(\overline{A}) \times p_{\overline{A}}(B) + p(\overline{A}) \times p_{\overline{A}}(\overline{B})$$
$$= \frac{2}{5} \times \frac{3}{4} + \frac{3}{5} \times \frac{2}{4} + \frac{3}{5} \times \frac{2}{4} = \frac{6}{20} + \frac{6}{20} + \frac{6}{20} = \frac{18}{20} = \frac{9}{10}$$